

# A semi-smooth Newton method for solving convex quadratic programming problem under simplicial cone constraint <sup>\*</sup>

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## Abstract

In this paper the simplicial cone constrained convex quadratic programming problem is studied. The optimality conditions of this problem consist in a linear complementarity problem. This fact, under a suitable condition, leads to an equivalence between the simplicial cone constrained convex quadratic programming problem and the one of finding the unique solution of a nonsmooth system of equations. It is shown that a semi-smooth Newton method applied to this nonsmooth system of equations is always well defined and under a mild assumption on the simplicial cone the method generates a sequence that converges linearly to its solution. Besides, we also show that the generated sequence is bounded for any starting point and a formula for any accumulation point of this sequence is presented. The presented numerical results suggest that this approach achieves accurate solutions to large problems in few iterations.

**Keywords:** Quadratic programming, convex set, convex cone, semi-smooth Newton method.

## 1 Introduction

The purpose of this paper is to motivate and describe a new approach for solving a special class of constrained convex quadratic programming problems, namely, simplicial cone constrained ones, by using the semi-smooth Newton's method, and to present the results of some computational experiments designed to investigate its practical viability.

Simplicial cone constrained convex quadratic programming arises as an important problem in its own right, it has an important subclass of positively constrained convex quadratic programming, namely, the positively constrained least-squares problems, or equivalently the problem of projecting the point onto a simplicial cone. The interest in the subject of projection arises in several situations,

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having a wide range of applications in pure and applied mathematics such as Convex Analysis (see e.g. [16]), Optimization (see e.g. [3, 6, 7, 15, 31, 34]), Numerical Linear Algebra (see e.g. [32]), Statistics (see e.g. [5, 11, 17]), Computer Graphics (see e.g. [14] ) and Ordered Vector Spaces (see e.g. [1, 19, 20, 27–29]). More specifically, the projection onto a polyhedral cone, which has as a special case the projection onto a simplicial one, is a problem of high impact on scientific community<sup>1</sup>. The geometric nature of this problem makes it particularly interesting and important in many areas of science and technology such as Statistics (see e.g. [17]), Computation (see e.g. [18]), Optimization (see e.g. [24, 34]) and Ordered Vector Spaces (see e.g. [27]).

The projection onto a general simplicial cone is difficult and computationally expensive, this problem has been studied e.g. in [2, 12, 15, 26, 27, 34]. It is a special convex quadratic program and its KKT optimality conditions consists in a linear complementarity problem (LCP) associated with it, see e.g [25, 26, 34]. Therefore, the problem of projecting onto simplicial cones can be solved by active set methods [4, 21, 22, 25] or any algorithms for solving LCPs, see e.g [4, 25] and special methods based on its geometry, see e.g [25, 26]. Other fashionable ways to solve this problem are based on the classical von Neumann algorithm (see e.g. the Dykstra algorithm [9, 11, 35]). Nevertheless, these methods are also quite expensive (see the numerical results in [24] and the remark preceding section 6.3 in [33]).

The KKT optimality conditions of simplicial cone constrained convex quadratic programming consist in a linear complementarity problem. Under a suitable condition, this leads to an equivalence between the simplicial cone constrained convex quadratic programming problem and the one of finding the unique solution of a nonsmooth system of equations. It is worth pointing out that a similar equation has been studied by Mangasaria in [23], which have used the semi-smooth Newton method for solving that equation, namely, an absolute value equation. Following the idea of [23], we apply the semi-smooth Newton’s method, see [30], to find a unique solution of the associated nonsmooth system of equations, which generates the solution of the simplicial cone constrained convex quadratic programming. Under a mild assumption on the simplicial cone we show that the method generates a sequence that converges linearly to the solution of the associated system of equations. This new approach has potential advantages over existing methods. The main advantage appears to be the ability to achieve accurate solutions to large problems in relatively few iterations. The global and linear convergence properties partially explain this good behavior. Our numerical results suggest that for a given problem class, the number of required iterations is almost unchanged. The numerical results also indicate a remarkable robustness with respect to the starting point.

The organization of the paper is as follows. In Section 1.1, some notations and basic results used in the paper are presented. In the beginning of Section 2 our main problem, the simplicial cone constrained convex quadratic programming problem, is presented. In Section 2.1 a semi-smooth equation is presented whose any solution generates a solution of our convex quadratic programming problem and an existence and uniqueness result of the solution for this semi-smooth equation is obtained. In Section 2.2 we state and prove a convergence theorem on the semi-smooth Newton method for finding the solution of the semi-smooth equation associated to the simplicial cone constrained convex quadratic programming problem. In Section 3 we present some computational tests.

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<sup>1</sup>see the popularity of the Wikimization page Projection on Polyhedral Cone at <http://www.convexoptimization.com/wikimization/index.php/Special:Popularpages>

## 1.1 Notations and auxiliary results

In this subsection we fix the notations and present some auxiliary results used throughout the paper. Let  $\mathbb{R}^n$  denote the  $n$ -dimensional Euclidean space and let  $\langle \cdot, \cdot \rangle$  be the canonical scalar product and  $\| \cdot \|$  be the norm generated by it. The  $i$ -th component of a vector  $x \in \mathbb{R}^n$  is denoted by  $x_i$  for every  $i = 1, \dots, n$ . Define the nonnegative orthant as

$$\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_i \geq 0, j = 1, \dots, n\}.$$

For  $x \in \mathbb{R}^n$ ,  $\text{sgn}(x)$  will denote a vector with components equal to 1, 0 or  $-1$  depending on whether the corresponding component of the vector  $x$  is positive, zero or negative. If  $a \in \mathbb{R}$  and  $x \in \mathbb{R}^n$ , then denote  $a^+ := \max\{a, 0\}$ ,  $a^- := \max\{-a, 0\}$  and  $x^+$ ,  $x^-$  and  $|x|$  the vectors with  $i$ -th component equal to  $(x_i)^+$ ,  $(x_i)^-$  and  $|x_i|$ , respectively. From the definitions of  $x^+$  and  $x^-$  it is easy to conclude that

$$x = x^+ - x^-, \quad x^+ \in \mathbb{R}_+^n, \quad x^- \in \mathbb{R}_+^n, \quad \langle x^+, x^- \rangle = 0, \quad \forall x \in \mathbb{R}^n. \quad (1)$$

**Remark 1** *It is well known that the projection onto a convex set is continuous and nonexpansive, see [16]. Since the projection of the point  $x \in \mathbb{R}^n$  onto the nonnegative orthant is  $x^+$ , we conclude that  $\|z^+ - w^+\| \leq \|z - w\|$ , for all  $z, w \in \mathbb{R}^n$ .*

The set of all  $m \times n$  matrices with real entries is denoted by  $\mathbb{R}^{m \times n}$  and  $\mathbb{R}^n \equiv \mathbb{R}^{n \times 1}$ . The matrix  $I$  denotes the  $n \times n$  identity matrix. If  $x \in \mathbb{R}^n$  then  $\text{diag}(x)$  will denote an  $n \times n$  diagonal matrix with  $(i, i)$ -th entry equal to  $x_i$ ,  $i = 1, \dots, n$ . For an  $M \in \mathbb{R}^{n \times n}$  consider the norm defined by  $\|M\| := \max_{x \neq 0} \{\|Mx\| : x \in \mathbb{R}^n, \|x\| = 1\}$ . This definition implies

$$\|Mx\| \leq \|M\|\|x\|, \quad \|LM\| \leq \|L\|\|M\|, \quad (2)$$

for any matrices  $L, M \in \mathbb{R}^{n \times n}$ .

**Lemma 1 (Banach's Lemma)** *Let  $E \in \mathbb{R}^{n \times n}$  and  $I$  the  $n \times n$  identity matrix. If  $\|E\| < 1$ , then  $E - I$  is invertible and  $\|(E - I)^{-1}\| \leq 1/(1 - \|E\|)$ .*

We will call a closed set  $K \subset \mathbb{R}^n$  with nonempty interior a *cone* if the following conditions hold:

- (i)  $\lambda x + \mu y \in K$  for any  $\lambda, \mu \geq 0$  and  $x, y \in K$ ,
- (ii)  $x, -x \in K$  implies  $x = 0$ .

Let  $K \subset \mathbb{R}^n$  be a cone. The *dual cone* of  $K$  is the following set

$$K^* := \{x \in \mathbb{R}^n \mid \langle x, y \rangle \geq 0, \forall y \in K\}.$$

The *simplicial cone* associated to a nonsingular matrix  $A \in \mathbb{R}^{n \times n}$  is defined by

$$A\mathbb{R}_+^n := \{Ax : x \in \mathbb{R}_+^n\}, \quad (3)$$

The following result follows from the definition of the dual of a cone. For a proof see for example [1].

**Lemma 2** *Let  $A$  be an  $n \times n$  nonsingular matrix. Then,*

$$(A\mathbb{R}_+^n)^* = (A^\top)^{-1}\mathbb{R}_+^n.$$

We will need the following result, for a proof combine Proposition 2A.3 with Theorem 2A.6 of [10].

**Theorem 1** *Let  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  be a differentiable convex function and  $K$  be a closed, convex cone in  $\mathbb{R}^n$ . Then  $\bar{x}$  is a solution of the problem*

$$\begin{aligned} & \text{Minimize } \varphi(x) \\ & \text{subject to } x \in K, \end{aligned}$$

*if and only if  $\bar{x}$  satisfies the following optimality conditions*

$$x \in K, \quad \nabla\varphi(x) \in K^*, \quad \langle \nabla\varphi(x), x \rangle = 0.$$

We end this section with the basic contraction mapping principle, its proof can be found in of [10] (see Theorem 1A.3 page 15).

**Theorem 2 (basic contraction mapping principle)** *Let  $\mathbb{X}$  be a complete metric space with metric  $\rho$  and let  $\phi : \mathbb{X} \rightarrow \mathbb{X}$ . Suppose that there exists  $\lambda \in [0, 1)$  such that  $\rho(\phi(x), \phi(y)) \leq \lambda\rho(x, y)$ , for all  $x, y \in \mathbb{X}$ . Then there exists a unique  $x \in \mathbb{X}$  such that  $\phi(x) = x$*

## 2 Quadratic programming under a simplicial cone constraint

In this section we will present a semi-smooth Newton method for solving a special class of quadratic programming problems, namely, quadratic programming problems under a simplicial cone constraint. The statement of such a problem is:

**Problem 1 (quadratic programming problem under a simplicial cone constraint)** *Let  $Q \in \mathbb{R}^{n \times n}$  be a symmetric positive definite matrix,  $b \in \mathbb{R}^n$ ,  $c \in \mathbb{R}$  and  $A \in \mathbb{R}^{n \times n}$  a nonsingular matrix. Find a solution  $u$  of the convex programming problem*

$$\begin{aligned} & \text{Minimize } \frac{1}{2}x^\top Qx + x^\top b + c \\ & \text{subject to } x \in A\mathbb{R}_+^n. \end{aligned} \tag{4}$$

Let us present an important particular case of Problem 1.

**Example 1** *Given  $A \in \mathbb{R}^{n \times n}$  a nonsingular matrix and  $z \in \mathbb{R}^n$ . The projection  $P_{A\mathbb{R}_+^n}(z)$  of the point  $z$  onto the cone  $A\mathbb{R}_+^n$  is defined by*

$$P_{A\mathbb{R}_+^n}(z) := \operatorname{argmin} \left\{ \frac{1}{2}\|z - y\|^2 : y \in A\mathbb{R}_+^n \right\}.$$

From the definition of the simplicial cone associated with the matrix  $A$  in (3), the problem of projecting the point  $z \in \mathbb{R}^n$  onto a simplicial cone  $A\mathbb{R}_+^n$  may be stated as the following positively constrained quadratic programming problem

$$\begin{aligned} & \text{Minimize } \frac{1}{2}\|z - Ax\|^2, \\ & \text{subject to } x \in \mathbb{R}_+^n. \end{aligned}$$

Hence, if  $v \in \mathbb{R}^n$  is the unique solution of this problem then we have  $P_{A\mathbb{R}_+^n}(z) = Av$ . The above problem is equivalent to the following nonnegatively constrained quadratic programming problem

$$\begin{aligned} & \text{Minimize } \frac{1}{2}x^\top \tilde{Q}x + x^\top \tilde{b} + \tilde{c} \\ & \text{subject to } x \in \mathbb{R}_+^n, \end{aligned} \tag{5}$$

by taking  $\tilde{Q} = A^\top A$ ,  $\tilde{b} = -A^\top z$  and  $\tilde{c} = z^\top z/2$ . The optimality condition for problem (5) implies that its solution can be obtained by solving the following linear complementarity problem

$$y - \tilde{Q}x = \tilde{b}, \quad x \geq 0, \quad y \geq 0, \quad \langle x, y \rangle = 0. \tag{6}$$

It is easy to establish that corresponding to each nonnegative quadratic problems (5) and each linear complementarity problems (6) associated to symmetric positive definite matrices, there are equivalent problems of projection onto simplicial cones. Therefore, the problem of projecting onto simplicial cones can be solved by active set methods [4, 21, 22, 25] or any algorithms for solving LCPs, see e.g [4, 25] and special methods based on its geometry, see e.g [25, 26]. Other fashionable ways to solve this problem are based on the classical von Neumann algorithm (see e.g. the Dykstra algorithm [9, 11, 35]). Nevertheless, these methods are also quite expensive (see the numerical results in [24] and the remark preceding section 6.3 in [33]).

In the next section we will show that Problem 1 can be solved by finding a solution of a special semi-smooth equation.

## 2.1 The semi-smooth equation associated to quadratic programming

In this section we present a semi-smooth equation whose any solution generates a solution of Problem 1.

**Problem 2 (semi-smooth equation)** Let  $Q \in \mathbb{R}^{n \times n}$  be a symmetric positive definite matrix,  $b \in \mathbb{R}^n$ ,  $c \in \mathbb{R}$  and  $A \in \mathbb{R}^{n \times n}$  a nonsingular matrix. Find a solution  $u$  of the semi-smooth equation

$$\left[ A^\top QA - I \right] x^+ + x + A^\top b = 0. \tag{7}$$

Next we apply Theorem 1 for showing that a solution of Problem 2 generates a solution of Problem 1.

**Proposition 1** If the vector  $u$  is a solution of Problem 2, then  $Au^+$  is a solution of Problem 1.

**Proof.** Note that from (1) we have  $u^+ - u = u^-$  for all  $u \in \mathbb{R}^n$ . Thus, if  $u \in \mathbb{R}^n$  is a solution of Problem 2, then

$$A^\top (QAu^+ + b) = u^-.$$

Since  $A$  is a nonsingular matrix and  $u^- \in \mathbb{R}_+^n$ , it follows from the last equality that

$$QAu^+ + b = (A^\top)^{-1}u^- \in (A^\top)^{-1}\mathbb{R}_+^n.$$

Hence, by using Lemma 2 and  $\langle u^-, u^+ \rangle = 0$ , the last inclusion easily implies that

$$QAu^+ + b \in (A\mathbb{R}_+^n)^*, \quad \langle QAu^+ + b, Au^+ \rangle = 0.$$

Therefore, as  $Au^+ \in A\mathbb{R}_+^n$ , applying Theorem 1 with  $K = A\mathbb{R}_+^n$  and  $\varphi(x) = x^\top Qx/2 + x^\top b + c$ , the desired result follows.  $\square$

**Proposition 2** *Let  $\lambda \in \mathbb{R}$ . If  $\|A^\top QA - I\| \leq \lambda < 1$  then Problem 2 has a unique solution.*

**Proof.** The Problem 2 has a solution if and only if the function  $\phi(x) = -[A^\top QA - I]x^+ - A^\top b$ , has a fixed point. From the definition of the function  $\phi$  and (13), it follows that for all  $x, y \in \mathbb{R}^n$  we have

$$\phi(x) - \phi(y) = \int_0^1 -[A^\top QA - I] \text{diag}(\text{sgn}((y + t(x - y))^+)) (x - y) dt.$$

Since  $\|\text{diag}(\text{sgn}((y + t(x - y))^+))\| < 1$  and  $\|A^\top QA - I\| < \lambda < 1$  for all  $t \in [0, 1]$ , the last equality implies that

$$\|\phi(x) - \phi(y)\| \leq \lambda \|x - y\|, \quad \forall x, y \in \mathbb{R}^n.$$

Hence  $\phi$  is a contraction. Therefore applying Theorem 2 with  $\mathbb{X} = \mathbb{R}^n$  and  $\rho = \|\cdot\|$  we conclude that  $\phi$  has precisely a unique fixed point and consequently Problem 2 has precisely a unique solution.  $\square$

## 2.2 Semi-smooth Newton method

In this section our goal is to state and prove a convergence theorem on the semi-smooth Newton method for finding the solution of Problem 2. We will first prove the well-definedness of the sequence generated by the semi-smooth Newton method. Then, under suitable conditions, the  $Q$ -linear convergence will be established. We also give a condition for the Newton method to finish in a finite number of iterations. Finally, we show that the semi-smooth sequence generated by the Newton method is bounded and we give a formula for any accumulation point of it. The statement of the main theorem is:

**Theorem 3** *Let  $Q \in \mathbb{R}^{n \times n}$  be a symmetric positive definite matrix,  $b \in \mathbb{R}^n$ ,  $c \in \mathbb{R}$  and  $A \in \mathbb{R}^{n \times n}$  a nonsingular matrix. Then, the sequences  $\{x_k\}$  generated by the semi-smooth Newton Method for solving Problem 2,*

$$x_{k+1} = - \left( [A^\top QA - I] \text{diag}(\text{sgn}(x_k^+)) + I \right)^{-1} A^\top b, \quad (8)$$

for  $k = 0, 1, \dots$ , is well defined for any starting point  $x_0 \in \mathbb{R}^n$ . Moreover, if

$$\left\| A^\top QA - I \right\| < 1/2, \quad \forall x \in \mathbb{R}^n, \quad (9)$$

then the sequence  $\{x_k\}$  converges  $Q$ -linearly to  $u \in \mathbb{R}^n$ , the unique solution of Problem 2, as follows

$$\|u - x_{k+1}\| \leq \frac{\|A^\top QA - I\|}{1 - \|A^\top QA - I\|} \|u - x_k\|, \quad k = 0, 1, \dots, \quad (10)$$

As a consequence,  $Au^+$  is the solution of the Problem 1.

Henceforward we assume that all assumptions in Theorem 3 hold. The *semi-smooth Newton method*, see [30], for solving the Problem 2, i.e., for finding the zero of the function

$$F(x) := \left[ A^\top QA - I \right] x^+ + x + A^\top b. \quad (11)$$

with starting point  $x_0 \in \mathbb{R}^n$ , is formally defined by

$$F(x_k) + V_k(x_{k+1} - x_k) = 0, \quad V_k \in \partial F(x_k), \quad k = 0, 1, \dots, \quad (12)$$

where  $V_k$  is any subgradient in  $\partial F(x_k)$  the Clarke generalized Jacobian of  $F$  at  $x_k$ . Note that  $S(x)$ , a subgradient in the Clarke generalized Jacobian of the function  $F$  at  $x$  (see Definition 2.6.1 on page 70 of [8]), is given by

$$S(x) := \left[ A^\top QA - I \right] \text{diag}(\text{sgn}(x^+)) + I \in \partial F(x), \quad x \in \mathbb{R}^n. \quad (13)$$

Since  $\text{diag}(\text{sgn}(x^+))x = x^+$  for all  $x \in \mathbb{R}^n$ , taking into account (11) and (13), we conclude that  $S(x)x = F(x) - A^\top b$ . Thus, taking  $V_k = S(x_k)$ , equation (12) becomes

$$S(x_k)x_{k+1} = -A^\top b, \quad k = 0, 1, \dots, \quad (14)$$

which is an equivalent definition of semi-smooth Newton sequence for solving the semi-smooth Problem 2, i.e., equation (8), which formally defines a sequence  $\{x_k\}$  with starting point  $x_0 \in \mathbb{R}^n$ . Hence, the sequence  $\{x_k\}$  defined in (8) will be called *semi-smooth Newton sequence* for finding the zero of the function  $F$  defined in (11), or equivalently for solving the semi-smooth Problem 2.

**Lemma 3** *The matrix  $S(x)$  defined in (13) is nonsingular for all  $x \in \mathbb{R}^n$ . As a consequence, the semi-smooth Newton sequence  $\{x_k\}$  is well-defined, for any starting point  $x_0 \in \mathbb{R}^n$ .*

**Proof.** Let  $x \in \mathbb{R}^n$ . To simplify the notations let  $D = \text{diag}(\text{sgn}(x^+))$ . Thus, the matrix in  $S(x)$  becomes

$$\left[ A^\top QA - I \right] D + I.$$

Let us suppose, by contradiction, that this matrix is singular, i.e, there exists  $u \in \mathbb{R}^n$  such that

$$\left( \left[ A^\top QA - I \right] D + I \right) u = 0, \quad u \neq 0.$$

It is straightforward to see that the last formula is equivalent to

$$A^\top QADu = (D - I)u, \quad u \neq 0. \quad (15)$$

Since the matrix  $Q$  is symmetric and positive definite, there exists a nonsingular matrix  $L \in \mathbb{R}^{n \times n}$  such that  $Q = LL^\top$ . Taking into account that  $D^2 = D$  and  $Q = LL^\top$ , the equality in equation (15) easily implies that

$$\|L^\top ADu\|^2 = \langle DA^\top QADu, u \rangle = \langle (D^2 - D)u, u \rangle = 0.$$

Thus we have  $L^\top ADu = 0$ . As  $Q = LL^\top$  and  $L^\top ADu = 0$ , equation (15) implies that  $(D - I)u = 0$ , or equivalently,  $Du = u$ . Hence

$$L^\top Au = L^\top ADu = 0, \quad u \neq 0.$$

But this contradicts the nonsingularity of  $A$ , since  $L$  is nonsingular. Therefore, the matrix  $S(x)$  is nonsingular for all  $x \in \mathbb{R}^n$  and the first part of the lemma is proven.

The proof of the second part of the lemma is an immediate consequence of the definition of the semi-smooth Newton sequence  $\{x_k\}$  in (8), the definition of  $S(x)$  in (13), and the first part of the lemma.  $\square$

**Lemma 4** *Let  $S(x)$  be as defined in (13). If  $\|A^\top QA - I\| < 1$  for all  $x \in \mathbb{R}^n$  then*

$$\|S(x)^{-1}\| \leq \frac{1}{1 - \|A^\top QA - I\|}, \quad \forall x \in \mathbb{R}^n.$$

**Proof.** To simplify the notation take  $S(x) = -(E - I)$ , where the matrix  $E$  is defined by

$$E = \left[ I - A^\top QA \right] \text{diag}(\text{sgn}(x^+)).$$

Since the diagonal matrix  $\text{diag}(\text{sgn}(x^+))$  has components equal to 1 or 0, the definition of  $E$  and the assumption  $\|A^\top QA - I\| < 1$  implies that

$$\|E\| \leq \|A^\top QA - I\| < 1.$$

Therefore, as  $S(x) = -(E - I)$ , combining the last inequality with Lemma 1 and the definition of  $E$ , the desired inequality follows.  $\square$

**Lemma 5** *Let  $F$  be the function defined in (11) and  $S(x)$  be the matrix defined in (13). Then the following inequality holds:*

$$\|S(x) - S(y)\| \leq \|A^\top QA - I\|, \quad \forall x, y \in \mathbb{R}^n.$$

As a consequence,

$$\|F(x) - F(y) - S(y)(x - y)\| \leq \|A^\top QA - I\| \|x - y\|, \quad \forall x, y \in \mathbb{R}^n.$$

**Proof.** Let  $x, y \in \mathbb{R}^n$ . The definition in (13) implies that

$$\|S(x) - S(y)\| = \|A^\top QA - I\| \|\text{diag}(\text{sgn}(x^+)) - \text{diag}(\text{sgn}(y^+))\| \leq \|A^\top QA - I\|,$$

which is the first inequality of the lemma. For proving the second inequality of the lemma, note that the definitions in (11) and (13) imply

$$F(x) - F(y) - S(y)(x - y) = \int_0^1 [S(y + t(x - y)) - S(y)](x - y) dt.$$

Therefore, the result follows by taking the norm in both sides of the last equality and using the first part of the lemma.  $\square$

Finally, we are ready to prove the main result, namely, Theorem 3.

**Proof of Theorem 3.** The well-definedness, for any starting point  $x_0 \in \mathbb{R}^n$ , follows from Lemma 3. Using Proposition 2, we conclude that under assumption (9) Problem 2 has unique solution  $u \in \mathbb{R}^n$ .

Let  $F$  be the function defined in (11) and  $S(x)$  be the matrix defined in (13). Since  $u \in \mathbb{R}^n$  is the solution of Problem 2 we have  $F(u) = 0$ , which together with definition of  $\{x_k\}$  in (8) implies

$$u - x_{k+1} = -S(x_k)^{-1} [F(u) - F(x_k) - S(x_k)(u - x_k)], \quad k = 0, 1, \dots$$

Using properties of the norm in (2), last equality implies

$$\|u - x_{k+1}\| \leq \|S(x_k)^{-1}\| \| [F(u) - F(x_k) - S(x_k)(u - x_k)] \|, \quad k = 0, 1, \dots$$

Combining Lemma 4 with the second part of Lemma 5, we conclude from the last equality that

$$\|u - x_{k+1}\| \leq \frac{\|A^\top QA - I\|}{1 - \|A^\top QA - I\|} \|u - x_k\|, \quad k = 0, 1, \dots \quad (16)$$

Since  $\|A^\top QA - I\| < 1/2$ , we have  $\|A^\top QA - I\| / (1 - \|A^\top QA - I\|) < 1$ . Therefore, the inequality in (16) implies that  $\{x_k\}$  converges Q-linearly, from any starting point, to the solution  $u$  of Problem 2. Hence the first part of the theorem is proven.

Since  $u \in \mathbb{R}^n$  is the solution of Problem 2 the second part of the theorem follows by using Proposition 1.  $\square$

The next proposition gives a condition for the Newton iteration (8) to finish in a finite number of steps.

**Proposition 3** *If in (8) it happens that  $\text{sgn}(x_{k+1}^+) = \text{sgn}(x_k^+)$ , then  $x_{k+1}$  is a solution of Problem 2 and  $Ax_{k+1}^+$  is the solution of the Problem 1.*

**Proof.** If  $\text{sgn}(x_{k+1}^+) = \text{sgn}(x_k^+)$  in equation (8), then it becomes

$$\left\{ \left[ A^\top QA - I \right] \text{diag}(\text{sgn}(x_{k+1}^+)) + I \right\} x_{k+1} = -A^\top b. \quad (17)$$

Since  $\text{diag}(\text{sgn}(x_{k+1}^+))x_{k+1} = x_{k+1}^+$ , the last equality yields

$$\left[ A^\top QA - I \right] x_{k+1}^+ + x_{k+1} = -A^\top b,$$

which implies that  $x_{k+1}$  is a solution of Problem 2 and, by using Proposition 1, it follows that  $Ax_{k+1}^+$  is the solution of the Problem 1.  $\square$

The next proposition shows that the semi-smooth Newton sequence  $\{x_k\}$ , defined in (8), is bounded and gives a formula for any accumulation point of it, without assuming condition (9).

**Proposition 4** *The semi-smooth Newton sequence  $\{x_k\}$ , defined in (8), is bounded from any starting point. Moreover, for each accumulation point  $\bar{x}$  of  $\{x_k\}$  there exists an  $\hat{x} \in \mathbb{R}^n$  such that*

$$\left( \left( A^\top QA - I \right) \text{diag}(\text{sgn}(\hat{x}^+)) + I \right) \bar{x} = -A^\top b. \quad (18)$$

*In particular, if  $\text{sgn}(\bar{x}^+) = \text{sgn}(\hat{x}^+)$ , then  $\bar{x}$  is a solution of Problem 2 and  $A\bar{x}^+$  is the solution of Problem 1.*

**Proof.** Suppose to the contrary that  $\{x_k\}$  is unbounded. Note that, as there are only finitely many vectors  $\text{sgn}(x_k^+)$  with coordinates 0 or 1, there exists a vector  $\tilde{x} \in \mathbb{R}^m$  and a subsequence  $\{x_{k_i}\}$  of  $\{x_k\}$  such that

$$\text{sgn}(x_{k_i}^+) \equiv \text{sgn}(\tilde{x}^+).$$

Now, since  $\{x_k\}$  is unbounded and the unit sphere is compact, there exists a vector  $v \in \mathbb{R}^m$  and a subsequence  $\{x_{k_j}\}$  of  $\{x_{k_i}\}$  such that

$$\lim_{j \rightarrow \infty} \|x_{k_j+1}\| = \infty, \quad \lim_{j \rightarrow \infty} \frac{x_{k_j+1}}{\|x_{k_j+1}\|} = v \neq 0. \quad (19)$$

Therefore, as  $\text{sgn}(x_{k_j}^+) = \text{sgn}(\tilde{x}^+)$  for all  $j$ , the definition of the semi-smooth Newton sequence  $\{x_k\}$  in (14) implies

$$\left( \left( A^\top QA - I \right) \text{diag}(\text{sgn}(\tilde{x}^+)) + I \right) \frac{x_{k_j+1}}{\|x_{k_j+1}\|} = -\frac{A^\top b}{\|x_{k_j+1}\|}, \quad j = 0, 1, 2, \dots$$

By tending with  $j$  to infinity in the above equality and by taking into account (19), it follows that

$$\left( \left( A^\top QA - I \right) \text{diag}(\text{sgn}(\tilde{x}^+)) + I \right) v = 0,$$

which contradicts the first part of the Lemma 3 since  $v \neq 0$ . Therefore, the sequence  $\{x_k\}$  is bounded, which proves the first part of the proposition.

For proving the second part of the proposition, let  $\bar{x}$  be an accumulation point of the sequence  $\{x_k\}$ . Then, since there are only finitely many vectors  $\text{sgn}(x_k^+)$  with coordinates 0 or 1, there exists a vector  $\hat{x} \in \mathbb{R}^m$  and a subsequence  $\{x_{k_j}\}$  of  $\{x_k\}$  such that

$$\lim_{j \rightarrow \infty} x_{k_j+1} = \bar{x}, \quad \text{sgn}(x_{k_j}^+) \equiv \text{sgn}(\hat{x}^+),$$

Since  $\text{sgn}(x_{k_j}^+) = \text{sgn}(\hat{x}^+)$  for all  $j$ , the definition of the semi-smooth Newton sequence  $\{x_k\}$  in (8) implies

$$\left( (A^\top QA - I) \text{diag}(\text{sgn}(\hat{x}^+)) + I \right) x_{k_{j+1}} = -A^\top b, \quad j = 0, 1, 2, \dots$$

Taking the limit in the last equality as  $k_j$  goes to  $\infty$ , the second part of the proposition follows.

Finally, for proving last part of the proposition, use the assumption  $\text{sgn}(\bar{x}^+) = \text{sgn}(\hat{x}^+)$  and (18) to obtain

$$\left( (A^\top QA - I) \text{diag}(\text{sgn}(\bar{x}^+)) + I \right) \bar{x} = -A^\top b.$$

Therefore, taking into account that  $\text{diag}(\text{sgn}(\bar{x}^+))\bar{x} = \bar{x}^+$  it is easy to conclude from the above equality that  $\bar{x}^+$  is a solution of Problem 2 and, by using Proposition 1, we obtain that  $A\bar{x}^+$  is the solution of the Problem 1, which conclude the proof of the proposition.  $\square$

### 3 Computational results

In this section we test our semi-smooth Newton method (8) to find solutions on generated random instances of Problem 2. We present two types of experiments. In one of them, we guarantee that for each test problem the hypotheses given in Theorem 3 are satisfied and in the other they are not.

All programs were implemented in MATLAB Version 7.11 64-bit and run on a 3.40GHz Intel Core i5 – 4670 with 8.0GB of RAM. All MATLAB codes and generated data of this paper are available in <http://orizon.mat.ufg.br/pages/34449-publications>.

All experiments are based on the following general considerations:

- In order to accurately measure the method's runtime for a problem, each one of the test problems was solved 10 times and the runtime data collected. Then, we defined the corresponding *method's runtime for a problem* as the median of these measurements.
- Let  $\text{Tol } X \in \mathbb{R}_+$  be a relative bound, we consider that the method converged to the solution and stopped the execution when, for some  $k$ , the condition

$$\|u - x_k\| < \text{Tol } X(1 + \|u\|),$$

is satisfied. If the previous stopping criteria are not met within 100 iterations, we declare that the method did not converge.

#### 3.1 When the hypotheses of Theorem 3 are satisfied

In this experiment, we studied the behavior of the method on sets of 100 randomly generated test problems of dimension  $n = 2000, 3000, 4000, 5000$ , respectively. Furthermore, we analyzed the influence of the initial point in the convergence of the method on 1000 randomly generated test problems of dimension  $n = 100$ . For each test problem in this experiment the hypotheses given in the Theorem 3 are satisfied, generating each of them as follows:

- (i) To construct the matrices  $A, Q \in \mathbb{R}^{n \times n}$  satisfying the assumption (9) in Theorem 3, we first chose a random number  $\beta$  from the standard uniform distribution on the open interval  $(0, 1/2)$ . Secondly, we compute the symmetric positive definite matrix  $Q = B^T B$ , where  $B$  is a generated  $n \times n$  real nonsingular matrix containing random values drawn from the uniform distribution on the interval  $[-10^6, 10^6]$ . Then, we compute the matrices  $S, V$  and  $D$ , respectively, from the singular value decomposition of a generated  $n \times n$  real nonsingular matrix containing random values drawn from the uniform distribution on the interval  $[-10^6, 10^6]$ . Finally, we compute the matrix  $A$  from the system of linear equations

$$BA = S \text{ sqrt} \left( I + \frac{\beta}{\sigma} V \right) D,$$

where  $\sigma$  is the largest singular value of  $V$  and  $\text{sqrt}(I + \frac{\beta}{\sigma} V)$  is the square root of the diagonal matrix  $I + \frac{\beta}{\sigma} V$ .

- (ii) We have chosen the solution  $u \in \mathbb{R}^n$  containing random values drawn from the uniform distribution on the interval  $[-10^6, 10^6]$  and then we have computed  $b \in \mathbb{R}^n$  from equation (7).
- (iii) Finally we have chosen a starting point  $x_0 \in \mathbb{R}^n$  containing random values drawn from the uniform distribution on the interval  $[-10^6, 10^6]$ .

In accordance with the theoretical convergence of the method, ensured by Theorem 3, the computational convergence is obtained in all cases.

The computational results to analyze the behavior of the method on sets of 100 generated random test problems of different dimensions, are reported in Table 1. From these, it can be noted that for the same dimension, to achieve higher accuracy, the method does not experience a significant increase in the number of iterations or in runtime. On the other hand, the increase in the dimension of the problems does not necessarily involve an increase in the number of iterations to achieve the same accuracy, however, a larger runtime is consumed. A larger runtime consumption is associated with the fact that the semi-smooth Newton method (8) requires the solution of a linear system in each iteration, whose computational effort increases with the dimension of the problem. Another important aspect that can be checked in Table 1 is the ability of the method to converge in about three iterations on average.

$n$	Total Iterations			Total Time		
2000	284	299	300	334.243	351.823	352.922
3000	279	293	295	1064.158	1117.909	1124.941
4000	281	303	303	2481.145	2676.010	2674.550
5000	283	303	305	4927.101	5261.154	5142.072
Tol $X$	$10^{-6}$	$10^{-8}$	$10^{-10}$	$10^{-6}$	$10^{-8}$	$10^{-10}$

Table 1: Total overall iterations and total time in seconds, performed and consumed, respectively by the semi-smooth Newton method (8) to solve the 100 test problems of each dimension for different accuracies.

In order to study the influence of the initial point in the convergence of the method, we have generated 1000 test problems of dimension  $n = 100$  and we have associated to each of them 1000 generated initial points. We have solved each problem with each of the 1000 corresponding initial points. Then, we have computed the standard deviation (STD)  $\bar{d}_i$  and the mean value (MEAN)  $\bar{m}_i$  of the number of iterations performed by the method to solve the problem  $i$  taking each one of the 1000 initial points. Finally we have computed the mean of all  $\bar{d}_i$  and the mean of all  $\bar{m}_i$ ,  $i = 1, \dots, 1000$ . All cases converged, indicating robustness of the method with respect to the starting point. The results are shown in Table 2. The standard deviation of the number of iterations performed by the method to solve the problem  $i$  with the 1000 initial points gives us an idea of the influence of the initial point in the number of iterations performed by the method in each problem. The reported means of these standard deviation values give us an idea of the influence of the initial point in the number of iterations performed by the method in all the problems in general. The results in the table show that on average the number of iterations performed by our method to find the solution for a problem varies only very slightly with the chosen starting point. Again we see that the average number of iterations performed is less than three.

Tol X	MEAN ( $\{\bar{d}_i\}_{i=1, \dots, 1000}$ )	MEAN ( $\{\bar{m}_i\}_{i=1, \dots, 1000}$ )
$10^{-6}$	0.241	2.337
$10^{-8}$	0.249	2.348
$10^{-10}$	0.249	2.348

Table 2: Influence of the initial point in the convergence of the semi-smooth Newton method (8) on a total of 1000 test problems of dimension  $n = 100$  each of them with 1000 generated initial points for different accuracies.

### 3.2 When the hypotheses of Theorem 3 are not satisfied

In this experiment, we studied the behavior of the method on 1000 test problems of dimension  $n = 100$ , where the hypotheses given in the Theorem 3 are not all satisfied.

In this case, the test problems were built almost as in the previous experiment. The only difference was in the construction of the matrices  $A, Q \in \mathbb{R}^{n \times n}$  not satisfying the assumption (9) of Theorem 3. Namely, we chose the random number  $\beta$  from the standard uniform distribution on the interval  $[lb, ub)$ , where  $\frac{1}{2} \leq lb < ub$ . Then,  $\|A^T Q A - I\| = \beta$ .

According to the obtained numerical results, we can conjecture that our method converges to a much broader class of problems, not satisfying the hypotheses of Theorem 3. However we detected that convergence with high accuracy to the solution largely depends on the magnitude of the value of the norm in condition (9). This idea can be observed inspecting Table 3. As the magnitude of the value of the norm in (9) increases, the number of problems for which the method converges decreases, and decreases the number of problems for which the method converges to the solution with greater accuracy. It can be also seen that for the same value of the norm in (9), the number of problems for which the method converges with greater precision reduces. This phenomenon, of course, is not associated to the convergence of the method for a specific problem, but, rather, there is an optimum accuracy achievable due to the accumulated errors. Small tolerances do not ensure

obtaining accurate results. It can be the case that convergence is overlooked and unnecessary iterations are performed. It is important to note in the table that, even when the hypothesis is unfulfilled, the method converges for these problems in an average of less than seven iterations, which means an increase of approximately four iterations with respect of the previous experiments in which the hypotheses were fulfilled.

$\beta \in [lb, ub)$	Solved Problems			Iterations		
$[0.5, 10^3)$	1000	1000	994	5.813	5.813	5.813
$[10^3, 10^4)$	1000	1000	966	6.318	6.318	6.316
$[10^4, 10^5)$	1000	995	539	6.389	6.389	6.455
$[10^5, 10^6)$	1000	964	3	6.436	6.438	6
$[10^6, 10^7)$	995	547	0	6.467	6.497	-
$[10^7, 10^8)$	960	3	0	6.436	6.667	-
Tol X	$10^{-6}$	$10^{-8}$	$10^{-10}$	$10^{-6}$	$10^{-8}$	$10^{-10}$

Table 3: Number of problems solved by the semi-smooth Newton method (8) on a total of 1000 test problems of dimension  $n = 100$  of each condition ( $lb \leq \|A^TQA - I\| < ub$ ) for different accuracies, and the mean number of iterations performed by the semi-smooth Newton method (8) to solve one problem in each case.

## 4 Conclusions

In this paper we studied a special class of convex quadratic programs, namely, simplicial cone constrained convex quadratic programming problems, which, via its optimality conditions, is reduced to finding the unique solution of a nonsmooth system of equations. Our main result shows that, under a mild assumption on the simplicial cone, we can apply a semi-smooth Newton method for finding a unique solution of the obtained associated nonsmooth system of equations and that the generated sequence converges linearly to the solution for any starting point. It would be interesting to see whether the used technique can be applied for solving more general convex programs.

Since the optimality condition of a simplicial cone constrained convex quadratic programming problem consists in a certain type of linear complementarity problem, which is equivalent to the problem of finding the unique solution of a nonsmooth system of equations, another interesting problem to address is to compare our semi-smooth Newton method with active set methods [4, 21, 22, 25].

This paper is a continuation of [13], where we studied the problem of projection onto a simplicial cone by using a semi-smooth Newton method. We expect that the results of this paper become a further step towards solving general convex optimization problems. We foresee further progress in this topic in the nearby future.

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