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Dini derivative and a characterization for Lipschitz and convex functions on Riemannian manifolds

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Abstract

Dini derivatives in Riemannian manifold settings are studied in this paper. In addition, a characterization for Lipschitz and convex functions defined on Riemannian manifolds and sufficient optimality conditions for constraint optimization problems in terms of the Dini derivative are given.

Keywords: Dini derivative; Convex functions; Lipschitz functions; Riemannian manifolds

1. Introduction

In the last few years, severals important concepts of nonsmooth analysis have been extended from Euclidean space to a Riemannian manifold setting, in order to go further in the study of optimization problems and related topics. Works dealing with this subject include those by Azagra, Ferrera and López-Mesas [1], Azagra, Ferrera [2], Ferreira [6] and Ledyaev and Zhu [9–11]. It is worthwhile to mention that extensions of concepts and techniques of optimization from Euclidean spaces to Riemannian manifolds have been extensively studied in several papers including [7,13–15,18].

Lipschitz and convex functions play an important role in nonsmooth analysis on linear spaces and, as is well known, the Dini derivative is a very useful tool in the analysis of these functions. Our aim in this paper is to study some properties of Dini derivatives in a Riemannian manifolds context and provide a characterization for Lipschitz and convex functions in terms of this derivative. In addition, we obtain sufficient optimality conditions for constraint optimization problems in that setting.

A couple of papers have dealt with the issue of characterization for Lipschitz and convex functions in the context of linear spaces. Clarke, Stern and Wolenski [4] have given a characterization for Lipschitz functions in terms of both the Dini derivative and proximal subgradients in the context of Hilbert space. Poliquin [16] provided a characterization for convex functions in terms of proximal subgradients on the Euclidean space $\mathbb{R}^n$ and Clarke, Stern and Wolenski [4] using a novel approach extended this result to Hilbert space. Correa, Jofré and Thibault [5] have given a characterization for convex functions in terms of Clarke subdifferentials in the context of Banach space. Also,

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Luc and Swaminathan [12] have obtained a characterization for convex functions in terms of Dini derivatives on real topological vector spaces.

The organization of the paper is as follows: In Section 1.1, we list some basic notations and auxiliary results used in this presentation. In Section 2 we introduce the concept of Dini derivatives of functions defined on Riemannian manifolds and obtain some results for it. In Section 3 we provide a characterization for Lipschitz functions in terms of Dini derivatives in a Riemannian manifold setting. In Section 4, we obtain a characterization for convex functions defined on Riemannian manifolds in terms of the Dini derivative. In Section 5 we give sufficient optimality conditions for constraint optimization problems in terms of the Dini derivative. We conclude this paper by making some remarks about extensions of our results.

1.1. Notation and auxiliary results

In this section we recall some notations, definitions and basic properties of Riemannian manifolds used throughout the paper. They can be found in many introductory books on Riemannian Geometry, for example in [17].

Throughout the paper, M is a smooth manifold and C^1 is the class of all continuously differentiable functions on M. The space of vector fields on M is denoted by $\mathcal{X}(M)$, by T_pM the tangent space of M at p and by $TM = \bigcup_{x\in M} T_xM$ the *tangent bundle* of M. Let M be endowed with a Riemannian metric $\langle , \rangle$, with corresponding norm denoted by $\| \ \|$, so that M is now a *Riemannian manifold*. Let us recall that the metric can be used to define the length of a piecewise C^1 curve $c : [a, b] \to M$ joining p to q, i.e., such that $c(a) = p$ and $c(b) = q$, by $l(c) = \int_a^b \|c'(t)\|\mathrm{d}t$. Minimizing this length functional over the set of all such curves we obtain a distance $d(p, q)$, which induces the original topology on M. Also, the metric induces a map $f \in C^1(M) \mapsto \operatorname{grad} f \in \mathcal{X}(M)$, which associates to each f its *gradient* via the rule $\langle \operatorname{grad} f, X\rangle = df(X)$, for all $X \in \mathcal{X}(M)$. The chain rule generalizes to this setting in the usual way: $(f \circ c)'(t) = \langle \operatorname{grad} f(c(t)), c'(t)\rangle$, for all curves $c \in C^1$. Let c be a curve joining points p and q in M and let ∇ be a Levi-Civita connection associated to $(M, \langle , \rangle)$. For each $t \in [a, b]$, ∇ induces an isometry, relative to $\langle , \rangle$, $P(c)_t^a : T_{c(a)}M \to T_{c(t)}M$, the so-called *parallel translation* along c from $c(a)$ to $c(t)$. A vector field V along c is said to be *parallel* if $\nabla_{c'} V = 0$. If c' itself is parallel, then we say that c is a *geodesic*. The geodesic equation $\nabla_{\gamma'}\gamma' = 0$ is a second order nonlinear ordinary differential equation, so the geodesic γ is determined by its position and velocity at one point. It is easy to check that $\|\gamma'\|$ is constant. We say that γ is *normalized* if $\|\gamma'\| = 1$. A geodesic $\gamma : [a, b] \to M$ is said to be *minimal* if its length is equal to the distance between its end points, i.e. $l(\gamma) = d(\gamma(a), \gamma(b))$.

A finite dimensional Riemannian manifold is *complete* if its geodesics are defined for any value of t. The Hopf–Rinow theorem asserts that if the Riemannian manifold M is complete, then any pair of points in M can be joined by a (not necessarily unique) minimal geodesic. Moreover, (M, d) is a complete metric space and its closed and bounded subsets are compact. *In this paper, we assume that all manifolds are complete* and *finite dimensional*.

The *exponential map* $\exp_p : T_pM \to M$ is defined by $\exp_p v = \gamma_v(1)$, where γ_v is the geodesic defined by its position p and velocity v at p. We can prove that, $\gamma_v(t) = \exp_p tv$ for any value of t. For $p \in M$, let

$$i_p := \sup\left\{ r > 0 : \exp_{p_{|B_r(o_p)}} \text{ is diffeomorphism} \right\},$$

where o_p denotes the origin of T_pM and $B_r(o_p) := \{v \in T_pM : \|v - o_p\| < r\}$. Note that if $0 < \delta < i_p$ then $\exp_p B_\delta(o_p) = B_\delta(p)$, where $B_\delta(p) := \{q \in M : d(p, q) < \delta\}$. The number i_p is called the *injetivity radius* of M at p. For any $p \in M$, the map $d^2(p, .) \in C^\infty(B_{i_p}(p))$, where $B_{i_p}(p) = \{q \in M : d(p, q) < i_p\}$, and $\operatorname{grad} d^2(p, q) = -2 \exp_q^{-1} p$, for all $q \in B_{i_p}(p)$. Throughout the paper, M denotes a finite dimensional Riemannian manifold which is complete and for any $p \in M$, for $\delta > 0$ we denote by $B_\delta(p)$ the open ball and by $B_\delta[p]$ the closed ball centered atn p.

The set $C \subseteq M$ is said to be *convex*, if for any $p, q \in C$ the minimal geodesics joining p to q are contained in C.

Let U be an open subset of M. From now on, we denote by $\mathcal{F}(U)$ the class of all function $f : M \to (-\infty, +\infty]$ which are lower semicontinuous on U and $\operatorname{dom}(f) \cap U \neq \emptyset$, where $\operatorname{dom}(f) = \{x \in M : f(x) < +\infty\}$. If $U = M$ we denote $\mathcal{F}$ for $\mathcal{F}(U)$.

2. Dini derivative

In this section we study some properties of the Dini derivative of locally Lipschitz functions. Our main result shows that the Dini derivative does not depend on the curve, namely, it just depends on the direction.

Definition 1. A function $f \in \mathcal{F}(U)$ is said to be Lipschitz on V, of rank $L \geq 0$, if $V \subset \text{dom}(f)$ and there holds

$$|f(p) - f(q)| \leqslant Ld(p, q), \quad \forall p, q \in V,$$

where d is the Riemannian distance on M.

We denote the set of all Lipschitz function on V, of rank L, by $\text{Lip}_L(V)$.

Definition 2. A function $f \in \mathcal{F}(U)$ is said to be Lipschitz at p, of rank L_p, if there exists $\delta > 0$ such that $f \in \text{Lip}_{L_p}(B_\delta(p))$ and locally Lipschitz on V if it is Lipschitz at every $p \in V$.

Now, we are going to present three important examples of Lipschitz functions that emerge in the study of Riemannian manifolds, see [17,8].

Example 1. Let M be a non-compact Riemannian manifold. A geodesic $\gamma : [0, +\infty) \to M$ parameterized by arc-length and emanating from p is called a ray emanating from p if $d(\gamma(t), \gamma(s)) = |t - s|$, for all $t, s > 0$. For a ray γ, the Busemann function $b_\gamma : M \to (-\infty, +\infty]$, defined by

$$b_\gamma(q) = \lim_{t \to +\infty} (t - d(q, \gamma(t))),$$

is Lipschitz of rank 1, see [6].

Example 2. Let M be a Riemannian manifold and S a closed subset of M. The distance function $d_S(p) = \inf\{d(p, s) : s \in S\}$ is Lipschitz of rank 1, see [6].

Example 3. Let $f \in \mathcal{F}$ and $\lambda > 0$. Suppose that f is bounded below by the constant k. Then, it easy to show that the function $f_\lambda : M \to (-\infty, +\infty]$ defined by $f_\lambda(p) = \inf\left\{f(q) + \lambda d^2(p, q)\right\}$, is also bounded below by k. Moreover, following the same pattern used to prove the first part of the Theorem 5.1 on page 44 of [3], we can prove that f_λ is locally Lipschitz on its domain.

We remark that the Lipschitz properties depend on the Riemannian metric defined on M. In other words, if the metric on M is changed then the set of Lipschitz functions on M becomes different from the previous one, see Example 4.4 in [6].

Definition 3. The lower Dini derivative of $f \in \mathcal{F}$ at $p \in \text{dom}(f)$ in the direction of $v \in T_pM$ is defined by

$$f'(p, v) = \liminf_{t \to 0^+} \frac{f(\gamma(t)) - f(p)}{t},$$

where $\gamma : \mathbb{R} \to M$ is the geodesic such that $\gamma(0) = p$ and $\gamma'(0) = v$. If $p \notin \text{dom}(f)$ define $f'(p, v) = -\infty$ for all $v \in T_pM$.

Note that if $f \in \mathcal{F}$ is a locally Lipschitz function on $\text{dom}(f)$, then $\text{dom}(f)$ is an open set and for all $p \in \text{dom}(f)$ and $v \in T_pM$ there exists the directional derivative and $f'(p, v) < +\infty$. Now, we are going to obtain some properties for lower Dini derivatives. We begin with some preliminaries.

Let $p \in M$ and $c_1, c_2 : (-\varepsilon, \varepsilon) \to M$ be a two C^1 curves, such that $c_1(0) = c_2(0) = p$, $c_1'(0) = v$ and $c_2'(0) = w$. Assume that the image sets $c_1((-\varepsilon, \varepsilon)), c_2((-\varepsilon, \varepsilon))$ are in $B_{i_p}(p)$. Let $\alpha : [0, 1] \times (-\varepsilon, \varepsilon) \to M$ be a *variation of geodesics* given by

$$\alpha(t, s) = \exp_{c_1(s)}(t \exp_{c_1(s)}{}^{-1} c_2(s)). \tag{1}$$

Note that, for each $s \in (-\varepsilon, \varepsilon)$ we have $\alpha(0, s) = c_1(s)$, $\alpha(1, s) = c_2(s)$ and the curve $\alpha_s : [0, 1] \to M$ given by $\alpha_s(t) = \alpha(t, s)$ is a geodesic. In particular, $\alpha_0(t) = \alpha(t, 0) = p$ is a constant geodesic. Now, consider the vector fields

$$T(\cdot, s) := \frac{\partial \alpha}{\partial t}(\cdot, s), \quad \text{and} \quad J(\cdot, s) = \frac{\partial \alpha}{\partial s}(\cdot, s).$$

Above definitions imply that $T(\cdot, s)$ is tangent to geodesic α_s and $J(\cdot, s)$, the *Jacobi vector field* through α_s, satisfies the following differential equation

$$\nabla_{\frac{\partial\alpha}{\partial t}}\nabla_{\frac{\partial\alpha}{\partial t}} J(t, s) + R(J(t, s), T(t, s))T(t, s) = 0, \tag{2}$$

where R is the curvature tensor, see [17].

Lemma 1. *Let $p \in M$, c_1 and c_2 be two C^1 curves in M such that $c_1(0) = c_2(0) = p$, $c_1'(0) = v$ and $c_2'(0) = w$. Then*

$$\lim_{s\to 0^+} \frac{d(c_1(s), c_2(s))}{s} = \|w - v\|.$$

Proof. To simplify notations, define $\psi(s) = d(c_1(s), c_2(s))$. Consider α, the variation of geodesics defined by (1), then $\psi(s) = \|\alpha_s'(t)\| = \|T(t, s)\|$. Since $T(t, 0) = \alpha'_0(t) = 0$ the first derivative of ψ^2 at $s = 0$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}s}(\psi^2(s))|_{s=0} = 2\left\langle \nabla_{\frac{\partial\alpha}{\partial s}} T(t, 0), T(t, 0)\right\rangle = 0, \tag{3}$$

and the second derivative by

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}s^2}(\psi^2(s))|_{s=0} &= 2\left\langle \nabla_{\frac{\partial\alpha}{\partial s}}\nabla_{\frac{\partial\alpha}{\partial s}} T(t, 0), T(t, 0)\right\rangle + 2\left\langle \nabla_{\frac{\partial\alpha}{\partial s}} T(t, 0), \nabla_{\frac{\partial\alpha}{\partial s}} T(t, 0)\right\rangle \\ &= 2\left\langle \nabla_{\frac{\partial\alpha}{\partial s}} T(t, 0), \nabla_{\frac{\partial\alpha}{\partial s}} T(t, 0)\right\rangle. \end{aligned} \tag{4}$$

In addition, Eq. (2) becomes

$$\nabla_{\frac{\partial\alpha}{\partial t}}\nabla_{\frac{\partial\alpha}{\partial t}} J(t, 0) = 0.$$

Now, the latter equality together with the conditions $J(0, 0) = v$ and $J(1, 0) = w$ implies that $J(t, 0) = v + t(w - v)$. Using Symmetry's Lemma (see Lemma 2.2, pp. 35 in [17]) and this equality we have

$$\nabla_{\frac{\partial\alpha}{\partial s}} T(t, 0) = \nabla_{\frac{\partial\alpha}{\partial s}} \frac{\partial\alpha}{\partial t}(t, 0) = \nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial s}(t, 0) = \nabla_{\frac{\partial\alpha}{\partial t}} J(t, 0) = w - v.$$

So, substituting the last equality in (4), we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}(\psi^2(s))|_{s=0} = 2\|w - v\|^2.$$

Thus, as $\psi(0) = 0$ it follows from the latter equality and (3) that $\psi^2(s) = \|w - v\|^2 s^2 + \mathcal{O}(s^2)$, with $\lim_{s\to 0^+} \mathcal{O}(s^2)/s^2 = 0$. Therefore, the result follows from the definition of ψ. □

Corollary 1. *If $f \in \mathcal{F}$ is Lipschitz in p with constant L_p, then $f'(p, .) \in \mathrm{Lip}_{L_p}(T_pM)$.*

Proof. Let v and w be in T_pM. Let γ, η be the geodesics with $\gamma(0) = \eta(0) = p$, $\gamma'(0) = v$ and $\eta'(0) = w$. First note that

$$\left| \frac{f(\gamma(t)) - f(p)}{t} - \frac{f(\eta(t)) - f(p)}{t} \right| = \frac{|f(\gamma(t)) - f(\eta(t))|}{t}.$$

Since f is Lipschitz in p and $\gamma(0) = \eta(0) = p$ we have $|f(\gamma(t)) - f(\eta(t))| \leqslant L_p d(\gamma(t), \eta(t))$ for all $t \in [0, \varepsilon)$ and some $\varepsilon > 0$. This inequality together with the above equality imply that

$$\frac{f(\gamma(t)) - f(p)}{t} - L_p \frac{d(\gamma(t), \eta(t))}{t} \leqslant \frac{f(\eta(t)) - f(p)}{t} \leqslant \frac{f(\gamma(t)) - f(p)}{t} + L_p \frac{d(\gamma(t), \eta(t))}{t},$$

for all $t \in [0, \varepsilon)$. Now, as $\gamma'(0) = v$ and $\eta'(0) = w$ we obtain from Lemma 1 that

$$\lim_{t\to 0^+} \frac{d(\gamma(t), \eta(t))}{t} = \|v - w\|.$$

Therefore, taking lim inf in the above inequality and considering the latter equality, we obtain

$$f'(p, v) - L_p\|v - w\| \leqslant f'(p, w) \leqslant f'(p, v) + L_p\|v - w\|,$$

which implies that $f'(x, .)$ is Lipschitz with constant L_p. □

Corollary 2. *Let $f \in \mathcal{F}$ be a Lipschitz function in $p \in \operatorname{dom}(f)$. For all C^1 curves $c : [0, \varepsilon) \to M$ satisfying $c(0) = p$ and $c'(0) = v$ there holds*

$$f'(p, v) = \liminf_{t\to 0^+} \frac{f(c(t)) - f(p)}{t}.$$

Proof. Let γ be a geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Since f is Lipschitz in p, a similar argument used in the proof of the latter corollary implies

$$\frac{f(\gamma(t)) - f(p)}{t} - L_p\frac{d(\gamma(t), c(t))}{t} \leqslant \frac{f(c(t)) - f(p)}{t} \leqslant \frac{f(\gamma(t)) - f(p)}{t} + L_p\frac{d(\gamma(t), c(t))}{t},$$

for all $t \in [0, \bar{\varepsilon})$ and some $\bar{\varepsilon} \leqslant \varepsilon$, where L_p is a Lipschitz constant of f in p. Now, as $c'(0) = \gamma'(0) = v$, Lemma 1 implies that $\lim_{t\to 0^+} d(\gamma(t), c(t))/t = 0$. So, taking lim inf in the latter inequality we obtain the statement. □

Corollary 3. *Let $f \in \mathcal{F}(U)$ be a locally Lipschitz function on U and $I \subset \mathbb{R}$ an open interval. If $c : I \to U$ is a C^1 curve, then*

$$(f \circ c)'(t, 1) = f'\left(c(t), c'(t)\right), \quad \forall t \in I.$$

Proof. This follows from Corollary 2. □

3. Characterization for Lipschitz functions

In this section we present a characterization for Lipschitz functions defined on Riemannian manifolds. It is worth pointing out that our results in this section were obtained by adapting, for our context, the techniques introduced by Clarke, Stern and Wolenski [4] for characterizing Lipschitz functions in Hilbert spaces.

Proposition 1. *Let U be an open convex subset of M and let $f \in \mathcal{F}(U)$. If f is locally Lipschitz on $\operatorname{dom} f$ with constant L everywhere, then the following statements hold:*

(i) $\operatorname{dom} f = U$;
(ii) $f \in \mathrm{Lip}_L(U)$.

Proof. For (i). First note that $\operatorname{dom}(f) \subset U$. It remains to show that $U \subset \operatorname{dom}(f)$. For that, take $q \in U$. Now, take $p \in \operatorname{dom}(f)$ and a minimal geodesic $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Since U is convex it follows that $\gamma([0, 1]) \subset U$. We claim that $q \in \operatorname{dom}(f)$. Suppose not. Thus we have that

$$t_* := \sup\{t \in (0, 1] : f(\gamma(t)) < +\infty\} < 1,$$

and since $p \in \operatorname{dom}(f)$ and f is locally Lipschitz on $\operatorname{dom} f$ we also have $0 < t_*$. So, taking $t' \in (0, t_*)$ we have $\gamma([0, t']) \subset \operatorname{dom} f$. Since $\gamma([0, t'])$ is compact and f is locally Lipschitz on $\operatorname{dom} f$, we can take $0 = t_0 < t_1 < \cdots < t_n = t'$ and positive numbers $\delta_0 \cdots \delta_{n-1}$ satisfying

$$\gamma([t_i, t_{i+1}]) \subset B_{\delta_i}(\gamma(t_i)) \subset U \quad \text{and} \quad f \in \mathrm{Lip}_L(B_{\delta_i}(\gamma(t_i))),$$

for all $i = 0, \ldots, n - 1$. As the geodesic γ is minimal, using the local Lipschitz property we obtain

$$\begin{aligned} f(\gamma(t')) &= f(p) + \sum_{i=0}^{n-1}(f(\gamma(t_{i+1})) - f(\gamma(t_i))) \\ &\leqslant f(p) + \sum_{i=0}^{n-1} L\, d(\gamma(t_i), \gamma(t_{i+1})) \\ &\leqslant f(p) + L\, d\left(p, \gamma(t')\right). \end{aligned} \tag{5}$$

Since f is lower semicontinuous, letting t' goes to $t*$ in the latter equation we conclude that

$$f(\gamma(t_*)) < +\infty,$$

which implies that $\gamma(t_*) \in \operatorname{dom}(f)$. According to f being Lipschitz at $\gamma(t_*)$, the definition of t_* is violated. Therefore, we obtain that f is finite on the entire $\gamma([0, 1])$. In particular, $q \in \operatorname{dom}(f)$, so $U \subset \operatorname{dom}(f)$ and the first statement follows.

For (ii). Take $p, q \in U$ and a minimal geodesic $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Since U is convex it follows from item (i) that $\gamma([0, 1]) \subset \operatorname{dom}(f)$. With an analogous argument used to obtain (5) we can show that

$$f(q) \leqslant f(p) + Ld(p, q).$$

Now, by reversing the roles of p and q, it easy to conclude that $f \in \mathrm{Lip}_L(U)$, and the second statement is proved. □

Theorem 1. *Let $f \in \mathcal{F}$ and U be an open convex subset of M. Then f is Lipschitz on U of rank $L \geq 0$ if, and only if,*

$$|f'(p, v)| \leqslant L\|v\|, \quad \forall p \in U, \forall v \in T_pM.$$

Proof. First, suppose that f is Lipschitz on U of rank $L \geq 0$. Take $p \in U$, $v \in T_pM$. Let γ be the geodesic such that $\gamma(0) = p$ and $\gamma'(0) = v$. Since $\gamma(0) = p$ and f is Lipschitz on U of rank $L \geq 0$, there exists $\delta > 0$ such that

$$|f(\gamma(t)) - f(p)| \leqslant Ld(\gamma(t), p) \leqslant L\|v\|, \quad \forall t \in [0, \delta).$$

So, using the definition of the lower Dini derivative, the above inequality implies that

$$|f'(p, v)| = \liminf_{t\to 0^+} \frac{|f(\gamma(t)) - f(p)|}{t} \leqslant L \liminf_{t\to 0^+} \frac{d(\gamma(t), p)}{t} = L\|v\|.$$

Reciprocally, suppose that $|f'(p, v)| \leqslant L\|v\|$, for all $p \in U$ and $v \in T_pM$. Let $p_0 \in \operatorname{dom}(f)$. Since $f \in \mathcal{F}(U)$, take $0 < \delta < i_p$ such that f is bounded below on $B_{4\delta}(p_0) \subset U$. Let $K > L$ and $q \in B_\delta(p_0)$. Define $g : U \to [0, +\infty]$ as

$$g(p) = \begin{cases} Kd(p, q) & \text{if } p \in B_{2\delta}[q]; \\ Kd(p, q) + \dfrac{(d(p, q) - 2\delta)^2}{3\delta - d(p, q)} & \text{if } p \in B_{3\delta}(q) \setminus B_\delta(q); \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that $g \in \mathcal{F}(U)$. Moreover, as $g \in C^1(B_{3\delta}(q) \setminus \{q\})$ and $0 < \delta < i_p$ there holds

$$\operatorname{grad} g(p) = \left[K + \frac{2(d(p, q) - 2\delta)(3\delta - d(p, q)) + d(p, q)(d(p, q) - 2\delta)}{(3\delta - d(p, q))^2}\right] \left(-{\exp_p}^{-1} q/d(p, q)\right),$$

for all $p \in B_{3\delta}(q) \setminus B_{2\delta}(q)$. Now, note that $f + g$ is in $\mathcal{F}(U)$, goes to $+\infty$ as p goes to the boundary of $B_{3\delta}(q)$ and is bounded below on $B_{3\delta}(q)$. Therefore, as M is a complete Riemannian manifold of finite dimension, there exists $p_* \in B_{3\delta}(q)$ a minimizer for $f + g$. First we assume that $p_* \neq q$. Since $p_* \in B_{3\delta}(q) \setminus \{q\}$, $g \in C^1(B_{3\delta}(q) \setminus \{q\})$ and $f + g \in \mathcal{F}(U)$ we have

$$f'(p_*, v) + \langle \operatorname{grad} g(p_*), v\rangle \geq 0 \quad \forall v \in T_{p_*}M. \tag{6}$$

For simplifying the notations set $v_* := \exp^{-1}_{p_*} q/d(p_*, q)$ and

$$\ell := \frac{2(d(p_*, q) - 2\delta)(3\delta - d(p_*, q)) + d(p_*, q)(d(p_*, q) - 2\delta)}{(3\delta - d(p_*, q))^2}.$$

So, $\operatorname{grad} g(p_*) = -Kv_*$ if $p_* \in B_{2\delta}[q]$ and $\operatorname{grad} g(p_*) = -[K + \ell]v_*$ if $p_* \in B_{3\delta}(q) \setminus B_{2\delta}(q)$. Thus, letting $v = v_*$ in (6) and taking into account that $\ell \geq 0$ for $p_* \in B_{3\delta}(q) \setminus B_{2\delta}(q)$ we obtain

$$f'(p_*, v_*) \geq K\|v_*\| > L\|v_*\|,$$

which contradicts our assumption. Consequently we must have $p_* = q$. Due to the fact that $g(q) = 0$, the point q is a minimizer of $f + g$ and $B_\delta(p_0) \subset B_{2\delta}[q]$ we have

$$f(q) = (f+g)(q) \leqslant (f+g)(p) \leqslant f(p) + Kd(p,q), \quad \forall p \in B_\delta(p_0).$$

Since we can change the roles of q and p in the above argument the following inequality holds

$$|f(q) - f(p)| \leqslant Kd(p,q), \quad \forall p, q \in B_\delta(p_0).$$

Letting K go to L in the latter inequality, we conclude that for any $p_0 \in \mathrm{dom} f$ there exists $\delta > 0$ such that $f \in \mathrm{Lip}_L B_\delta(p_0)$. Thus we have shown that f is locally Lipschitz on $\mathrm{dom}(f)$ with the same rank L everywhere. Therefore, for finishing the proof use the Proposition 1. □

Example 4. Let $S\mathbb{R}^n$ be the set of symmetric matrices endowed with the Frobenius metric defined by $\langle U, V\rangle = \mathrm{tr}\,(VU)$ and let $S\mathbb{R}^n_{++}$ be the set of positive definite symmetric matrices. Let the function $f : S\mathbb{R}^n_{++} \to \mathbb{R}$ be defined by $f(X) = \ln\det X$. It easy to see that the function f is not Lipschitz on $S\mathbb{R}^n_{++}$. For each $X \in S\mathbb{R}^n_{++}$ define a new inner product in $S\mathbb{R}^n$ as

$$\langle\langle U, V\rangle\rangle = \langle X^{-1}UX^{-1}, V\rangle \quad \forall U, V \in S\mathbb{R}^n.$$

Endowing $S\mathbb{R}^n_{++}$ with the Riemannian metric $\langle\langle\,,\ .\rangle\rangle$ we obtain a complete Riemannian manifold. We denote by M this Riemannian manifold. Note that $f \in C^1$ on M. So,

$$f'(X, V) = \langle\langle \mathrm{grad}\, f(X), V\rangle\rangle \quad \forall V \in T_X M.$$

Because the usual gradient of f on $S\mathbb{R}^n_{++}$ is $\nabla f(X) = X^{-1}$, we have that the gradient of f on M is given by

$$\mathrm{grad}\, f(X) = X\nabla f(X)X = X.$$

So, $\|\mathrm{grad}\, f(X)\|^2 = \langle\langle \mathrm{grad}\, f(X), \mathrm{grad}\, f(X)\rangle\rangle = 1$. Thus,

$$|f'(X, V)| = |\langle \mathrm{grad}\, f(X), V\rangle| \leqslant \|\mathrm{grad}\, f(X)\|\|V\| = \|V\|, \quad \forall\, V \in T_X M.$$

From Theorem 1 it follows that f is Lipschitz on M of rank $L = 1$.

Example 5. Let $\Omega = \{p = (p_1, p_2) \in \mathbb{R}^2 : p_2 > 0\}$ and let $f : \Omega \to \mathbb{R}$ be given by $f(p) = \ln(p_2)$. It easy to see that f is not a Lipschitz function on Ω with respect to the Euclidean metric $\langle,\rangle$. Let G be a 2×2 matrix defined by $G(p) = (g_{ij}(p))$, where

$$g_{11}(p) = g_{22}(p) = \frac{1}{p_2^2}, \qquad g_{12}(p) = g_{21}(p) = 0.$$

Endowing Ω with the Riemannian metric $\langle\langle,\rangle\rangle$ defined by $\langle\langle u, v\rangle\rangle = \langle G(p)v, u\rangle$, we obtain a complete Riemannian manifold $\mathbb{H}^2$, namely, the upper half-plane model of Hyperbolic space. Note that $f \in C^1$ and the gradient of f in $\mathbb{H}^2$ is given by

$$\mathrm{grad}\, f(p) = G(p)^{-1}\nabla f(p) = (0, p_2),$$

where ∇f is the usual gradient of f in Ω. It is simple to show that $\|\mathrm{grad}\, f(p)\| = 1$ and so,

$$|f'(p, v)| = |\langle \mathrm{grad}\, f(p), v\rangle| \leqslant \|v\|, \quad \forall\, v \in T_p\mathbb{H}^2.$$

Therefore, from Theorem 1 it follows that f is Lipschitz on $\mathbb{H}^2$ of rank $L = 1$.

4. Characterization for convex functions

In this section we obtain a characterization for convex functions defined on Riemannian manifolds. As usual, in the type of characterization we shall present, first we obtain a result like the mean value theorem adapted to our setting.

Definition 4. A function $f : M \to (-\infty, +\infty]$ is said to be convex (respectively, strictly convex) if for all minimal geodesics $\gamma : [a, b] \to M$, the composition $f \circ \gamma : [a, b] \to (-\infty, +\infty]$ is convex (respectively, strictly convex).

It follows from the above definition that if $f : M \to (-\infty, +\infty]$ is a convex function then $\mathrm{dom}(f)$ and the sub-level sets $\{p \in M : f(p) \le k\}$ are convex sets, for all $k \in \mathbb{R}$.

Example 6. Let $S^n = \{p \in \mathbb{R}^{n+1} : \|p\| = 1\}$ be a unitary sphere. Fix $\tilde{p} = (0, \dots, 0, 1) \in R^{n+1}$. Setting $p = (p_1 \dots, p_n, p_{n+1})$, define $\varphi : S^n \to (-\infty, +\infty]$ as

$$\varphi(p) = \begin{cases} d(\tilde{p}, p), & \text{if } p_{n+1} \ge 0; \\ +\infty & \text{if } p_{n+1} < 0. \end{cases}$$

Note that $\varphi \in \mathcal{F}$, but φ is not a convex function and its $\mathrm{dom}(\varphi) = \{p \in S^n : p_{n+1} \ge 0\}$ is not a convex set. Now, let $C \subset \{p \in S^n : p_{n+1} > 0\}$ be a closed and convex set and define $\rho : S^n \to (-\infty, +\infty]$ as

$$\rho(p) = \begin{cases} d(\tilde{p}, p), & \text{if } p \in C; \\ +\infty & \text{if } p \notin C. \end{cases}$$

Note that ρ is lower semicontinuous, convex and its domain C is closed. In general, for all $\tilde{p} \in M$ and $C \subset \{p \in M : d(\tilde{p}, p) < \pi/2\}$ a closed convex set, the function $\eta : S^n \to (-\infty, +\infty]$ defined as $\eta(p) = d(\tilde{p}, p)$, if $p \in C$ and $\eta(p) = +\infty$ if $p \notin C$, is lower semicontinuous and convex.

In the above example $\mathrm{dom}(\varphi) = \{p \in \mathbb{R}^n : p_{n+1} \ge 0\}$ is closed and its interior is convex. As $\mathrm{dom}(\varphi)$ is not convex we conclude that, in general, the closure of a convex set is not convex. Note that φ is Lipschitz of rank $L = 1$ in $\mathrm{int}(\mathrm{dom}(\varphi))$, but not in $\mathrm{dom}(\varphi)$.

Definition 5. A function $f \in \mathcal{F}$ is locally bounded in p if there exist $\delta > 0$ and $r > 0$ such that $f(q) \le r$ for all $p \in B_\delta(p)$, and $f \in \mathcal{F}$ is locally bounded in $\mathrm{dom}(f)$ if it is locally bounded at all points $p \in \mathrm{dom}(f)$.

Note that if $f \in \mathcal{F}$ is locally bounded in $\mathrm{dom}(f)$ then $\mathrm{dom}(f)$ must be open.

Proposition 2. *Let $f \in \mathcal{F}$ be locally bounded in* $\mathrm{dom}(f)$. *If f is convex, then f is locally Lipschitz on* $\mathrm{dom}(f)$.

Proof. See Proposition 5.2 in [1]. □

We remark that in n-dimensional Euclidean spaces for proving that a convex function f is locally bounded in $\mathrm{dom}(f)$ two important results are used, namely, for each $p \in \mathrm{dom}(f)$ there exists a n-dimensional simplex $\Delta \subset \mathrm{dom}(f)$ such that $p \in \mathrm{int}(\Delta)$ and Jensen's inequality. However, as far as we know results like these in the Riemannian context have not been studied yet.

From now on, we assume that $f \in \mathcal{F}$ and is locally Lipschitz on $\mathrm{dom}(f)$ without explicitly mentioning them in the statements of our results. Note that in this case f is locally bounded in $\mathrm{dom}(f)$ and $\mathrm{dom}(f)$ is open.

Example 7. Let $S^n = \{p \in \mathbb{R}^{n+1} : \|p\| = 1\}$ be a unitary sphere. Fix $\tilde{p} = (0, \dots, 0, 1) \in R^{n+1}$. Setting $p = (p_1 \dots, p_n, p_{n+1})$, define $\zeta : S^n \to (-\infty, +\infty]$ as

$$\zeta(p) = \begin{cases} -\ln(\pi/2 - d(\tilde{p}, p)), & \text{if } p_{n+1} > 0; \\ +\infty & \text{if } p_{n+1} < 0. \end{cases}$$

ζ is lower semicontinuous, convex and its domain is open. Moreover, ζ locally Lipschitz in $\mathrm{dom}(\zeta)$.

Example 8. Let $M = \{p \in \mathbb{R}^n : \|p\| = 1\}$ be a unitary sphere. Setting $p = (p_1 \dots, p_n)$, define $\psi : M \to (-\infty, +\infty]$ as

$$\psi(p) = \begin{cases} -\sum_{i=1}^{n} \ln(p_i) & \text{if } p_1 > 0, \dots, p_n > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

ψ is lower semicontinuous, convex and its domain is open. Moreover, ψ is locally Lipschitz in $\mathrm{dom}(\psi)$.

Proposition 3. *If f is convex, then for all $p \in \mathrm{dom}(f)$ and $v \in T_pM$ there holds*

$$f'(p, v) = \lim_{t\to 0^+} \frac{f(\gamma(t)) - f(p)}{t} = \inf_{t>0} \frac{f(\gamma(t)) - f(p)}{t},$$

where γ is a minimal geodesic such that $\gamma(0) = p$ and $\gamma'(0) = v$.

Proof. It follows from convexity of $f \circ \gamma$. □

Corollary 4. *If f is convex and $\gamma : [a, b] \to \mathrm{dom}(f)$ is a minimal geodesic, then there holds*

$$(t - \tilde{t}) f'(\gamma(\tilde{t}), \gamma'(\tilde{t})) + f(\gamma(\tilde{t})) \leqslant f(\gamma(t)), \quad \forall\, \tilde{t},\ t \in [a, b].$$

Proof. Is an immediate consequence of Proposition 3. □

Definition 6. Let $f \in \mathcal{F}$ be a locally Lipschitz function on $\mathrm{dom}(f)$. The lower Dini derivative $f' : TM \to [-\infty, +\infty]$ is said to be monotone (respectively, strictly monotone) if, for any minimal geodesic $\gamma : [a, b] \to M$ with its end points in $\mathrm{dom}(f)$, the map $\varphi_\gamma : [a, b] \to [-\infty, +\infty]$ defined by $\varphi_\gamma(t) = f'(\gamma(t), \gamma'(t))$ is monotone non-decreasing (respectively, increasing).

Note that in the above definition we do not assume that $\mathrm{dom}(f)$ is a convex set.

Lemma 2. *If γ is a geodesic such that $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma([0,\ 1]) \subset \mathrm{dom}(f)$, then there exist $\tilde{t}, \hat{t} \in (0, 1)$ such that*

$$f'\left(\gamma(\tilde{t}), \gamma'(\tilde{t})\right) \leqslant f(q) - f(p) \leqslant f'\left(\gamma(\hat{t}), \gamma'(\hat{t})\right).$$

Proof. Let $\psi : [0, 1] \to R$ be defined by $\psi(s) = f(\gamma(s)) - s(f(q) - f(p))$. So, from Corollary 3 is sufficient to show that there exist $\tilde{t}, \hat{t} \in (0, 1)$ such that

$$\psi'(\tilde{t}, 1) \leqslant 0 \leqslant \psi'(\hat{t}, 1).$$

First we will prove that there exists $\tilde{t} \in (0, 1)$ such that $\psi'(\tilde{t}, 1) \leqslant 0$. Since ψ is continuous in $[0, 1]$ and $\psi(0) = \psi(1) = f(p)$, there exists $s_1 \in (0, 1]$ a global minimum of ψ. Now let $s_0 \in (0, s_1)$. We have two possibilities: (i) $\psi'(s_0, 1) \leqslant 0$ or (ii) $\psi'(s_0, 1) > 0$. If (i) occurs take $\tilde{t} = s_0$, otherwise $\psi'(s_0, 1) > 0$ implies that there exists $\bar{s} \in (s_0, s_1)$ such that $\psi(s_0) < \psi(\bar{s})$. Hence, by the continuity of ψ there exists $\tilde{t} \in (s_0, s_1)$ such that $\psi(s) \leqslant \psi(\tilde{t})$ for all $s \in [s_0, s_1]$, which implies $\psi'(\tilde{t}, 1) \leqslant 0$. The second inequality can be shown by a similar argument to the first. □

Corollary 5. *If γ is a geodesic in M such that $t_2 > t_1$, $\gamma([t_1, t_2]) \subset \mathrm{dom}(f)$, then there exist $\tilde{t}, \hat{t} \in (t_1, t_2)$ such that*

$$f'\left(\gamma(\tilde{t}), \gamma'(\tilde{t})\right) \leqslant \frac{f(\gamma(t_2)) - f(\gamma(t_1))}{t_2 - t_1} \leqslant f'\left(\gamma(\hat{t}), \gamma'(\hat{t})\right).$$

Proof. Defining the geodesic $\alpha(s) = \gamma(st_2 + (1 - s)t_1)$ we have $\alpha(0) = \gamma(t_1)$ and $\alpha(1) = \gamma(t_2)$. Now, note that $v \mapsto f'(p, v)$ is positively homogeneous, for all $p \in \mathrm{dom}(f)$. So, applying Lemma 2 the statement follows. □

Lemma 3. *If f' is monotone, then $\mathrm{dom}(f)$ is convex.*

Proof. Take $p, q \in \mathrm{dom}(f)$ and γ a minimal geodesic joining p to q. Assume without lose of generality that $\gamma(0) = p$ and $\gamma(1) = q$. Define

$$\hat{t} := \inf\{t \in [0, 1] : [\gamma(t),\ q] \in \mathrm{dom}(f)\}, \qquad \tilde{t} := \sup\{t \in [0, 1] : [p, \gamma(t)] \in \mathrm{dom}(f)\}.$$

Since f is locally Lipschitz on $\mathrm{dom}(f)$ we have that $\mathrm{dom}(f)$ is open, so $0 \leqslant \hat{t} < 1$ and $0 < \tilde{t} \leqslant 1$. Assume by contraction that $\gamma([0,\ 1]) \not\subseteq \mathrm{dom}(f)$ or equivalently that $0 < \hat{t}$ and $\tilde{t} < 1$. Now, take $0 < \lambda < \tilde{t} \leqslant \hat{t} < \tau < 1$. The definitions of $\hat{t}$ and $\tilde{t}$ imply that the minimal geodesic segments joining $\gamma(\tau)$ to q and p to $\gamma(\lambda)$ are in $\mathrm{dom}(f)$. Thus, from Corollary 5 there exist $t_1 \in (0, \lambda)$ and $t_2 \in (\tau, 1)$ such that

$$f(\gamma(\lambda)) - f(p) \leqslant \lambda f'(\gamma(t_1), \gamma'(t_1)), \qquad (1 - \tau) f'(\gamma(t_2), \gamma'(t_2)) \leqslant f(q) - f(\gamma(\tau)).$$

The above inequalities together with monotonicity of the derivative f' imply that

$$\frac{f(\gamma(\lambda))}{\lambda} + \frac{f(\gamma(\tau))}{(1-\tau)} \leqslant \frac{f(p)}{\lambda} + f'(\gamma(t_1), \gamma'(t_1)) + \frac{f(q)}{(1-\tau)} - f'(\gamma(t_2), \gamma'(t_2)) \leqslant \frac{f(p)}{\lambda} + \frac{f(q)}{(1-\tau)}.$$

Since $f \in \mathcal{F}$, $0 < \tilde{t}$ and $\hat{t} < 1$, letting λ go to $\tilde{t}$ and τ go to $\hat{t}$ in the above equation we obtain

$$\frac{f(\gamma(\tilde{t}))}{\tilde{t}} + \frac{f(\gamma(\hat{t}))}{(1-\hat{t})} \leqslant \frac{f(p)}{\tilde{t}} + \frac{f(q)}{(1-\hat{t})}.$$

So, $f(\gamma(\hat{t})) < +\infty$ and $f(\gamma(\tilde{t})) < +\infty$, i.e., $\gamma(\hat{t}), \gamma(\tilde{t}) \in \mathrm{dom}(f)$. As $\mathrm{dom}(f)$ is open we obtain a contradiction with the definitions of $\hat{t}$ and $\tilde{t}$. Therefore, $\gamma([0,\ 1]) \subseteq \mathrm{dom}(f)$ and the statement follows. □

Theorem 2. *f is convex if, and only if, f' is monotone.*

Proof. Assume that f is convex. First note that $\mathrm{dom}(f)$ is convex. Let $\gamma : [a, b] \to M$ be a minimal geodesic with its end points in $\mathrm{dom}(f)$ and take $t_1, t_2 \in [a, b]$ such that $t_1 < t_2$. Since $\mathrm{dom}(f)$ is convex we have that $\gamma([t_1, t_2]) \subseteq \mathrm{dom}(f)$ and using Corollary 4 we obtain that

$$(t_2 - t_1) f'\left(\gamma(t_1), \gamma'(t_1)\right) + f(\gamma(t_1)) \leqslant f(\gamma(t_2)), \qquad (t_1 - t_2) f'\left(\gamma(t_2), \gamma'(t_2)\right) + f(\gamma(t_2)) \leqslant f(\gamma(t_1)).$$

Therefore, as $t_1 < t_2$ it follows from last two inequalities that $f'\left(\gamma(t_1), \gamma'(t_1)\right) \leqslant f'\left(\gamma(t_2), \gamma'(t_2)\right)$, hence f' is monotone.

Now, assume that f' is monotone. Let $\gamma : [a, b] \to M$ be a geodesic and let $t_1, t_2 \in [a, b]$ be such that $t_1 < t_2$. If $\gamma(t_1) \notin \mathrm{dom}(f)$ or $\gamma(t_2) \notin \mathrm{dom}(f)$ we have

$$f \circ \gamma((1-\lambda)(t_1 + \lambda t_2)) \leqslant (1-\lambda) f(\gamma(t_1)) + \lambda f(\gamma(t_2)),$$

for all $\lambda \in [0, 1]$. Now, suppose that $\gamma(t_1), \gamma(t_2) \in \mathrm{dom}(f)$. So, Lemma 3 implies that $\gamma([t_1, t_2]) \subseteq \mathrm{dom}(f)$. Thus, for all $t \in (t_1, t_2)$ we obtain from Corollary 5 that there exist $s_1 \in (t_1, t)$ and $s_2 \in (t, t_2)$ such that

$$f(\gamma(t)) - f(\gamma(t_1)) \leqslant (t - t_1) f'(\gamma(s_1), \gamma'(s_1)), \qquad (t_2 - t) f'(\gamma(s_2), \gamma'(s_2)) \leqslant f(\gamma(t_2)) - f(\gamma(t)).$$

Thus, letting $t = (1-\lambda)t_1 + \lambda t_2$ in the above equation we obtain from monotonicity of f' that

$$\begin{aligned}(1-\lambda) f(\gamma(t_1)) + \lambda f(\gamma(t_2)) - f(\gamma(t)) &= -(1-\lambda)[f(\gamma(t)) - f(\gamma(t_1))] + \lambda[f(\gamma(t_2)) - f(\gamma(t))] \\ &\geq (1-\lambda)\lambda(t_2 - t_1)[-f'(\gamma(s_1), \gamma'(s_1)) + f'(\gamma(s_2), \gamma'(s_2))] \\ &\geq 0,\end{aligned}$$

for all $\lambda \in [0, 1]$. Which implies that $f \circ \gamma((1-\lambda)t_1 + \lambda t_2) \leqslant (1-\lambda) f(\gamma(t_1)) + \lambda f(\gamma(t_2))$, for all $\lambda \in [0, 1]$. Therefore f is convex. □

5. Sufficient optimality conditions for optimization problems

In this section we are going to obtain sufficient optimality conditions for constraint optimization problems in terms of the Dini derivative.

Proposition 4. *Let $C \subset M$ be a convex set. Assume that $f : M \to \mathbb{R}$ is convex in C. Consider the following nonlinear programming problem*

$$(P) \begin{cases} \min & f(p) \\ s.t. & p \in C. \end{cases}$$

Let $p^* \in C$. If for all $p \in C$ we have that $f'(p^*, \gamma'_{p^*p}(0)) \geq 0$, where γ_{p^*p} is a minimal geodesic from p^* to p with $\gamma_{p^*p}(0) = p^*$ and $\gamma_{p^*p}(1) = p$. Then $p^* \in C$ is a solution to (P).

Proof. Given $p, p^* \in C$ and γ_{p^*p} a minimal geodesic from p^* to p with $\gamma_{p^*p}(0) = p^*$ and $\gamma_{p^*p}(1) = p$. Since C is convex we have that $\gamma_{p^*p}([0, 1]) \subset C$. Now, as f is convex we conclude from Corollary 4 that

$$f(p) \geq f(p^*) + f'(p^*, \gamma'_{p^*p}(0)).$$

Because $f'(p^*, \gamma'_{p^*p}(0)) \geq 0$, the latter inequality implies that $f(p) \geq f(p^*)$. So, p^* is a solution to (P). □

Corollary 6. *Let $f, g_i : M \to \mathbb{R}$ be convex, for $i = 1, \ldots, m$. Consider the following nonlinear programming problem*

$$(\tilde{P}) \begin{cases} \min & f(p) \\ s.t. \; g_i(p) \leqslant 0, & i = 1, \ldots, m. \end{cases}$$

Let p^* be a feasible point to $(\tilde{P})$. If for all p, a feasible point to $(\tilde{P})$, there exists a vector $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m$ such that

$$f'(p^*, \gamma'_{p^*p}(t_*)) + \sum_{i=1}^{m} \mu_i g_i{}'(p^*, \gamma'_{p^*p}(t_*)) \geq 0, \quad \mu \geqslant 0, \quad and \quad \sum_{i=1}^{m} \mu_i g_i(p^*) = 0, \tag{7}$$

where γ_{p^*p} is a minimal geodesic from p^* to p with $\gamma'_{p^*p}(0) = p^*$ and $\gamma'_{p^*p}(1) = p$. Then p^* is a solution to $(\tilde{P})$.

Proof. Since $f, g_i : M \to \mathbb{R}$ are convex, for $i = 1, \ldots, m$, and $\mu \geqslant 0$ we conclude that $C := \{p \in M : g_i(p) \leqslant 0, i = 1, \ldots, m\}$ is convex and $h : M \to \mathbb{R}$ defined by $h(p) = f(p) + \sum_{i=1}^{m} \mu_i g_i(p)$ is also convex. Moreover,

$$f(p) \geq h(p), \quad \text{for all } p \in C. \tag{8}$$

Take $p \in C$ and γ_{p^*p} a minimal geodesic from p^* to p with $\gamma_{p^*p}(0) = p^*$ and $\gamma_{p^*p}(1) = p$. From the first inequality in (7) we obtain that $h'(p^*, \gamma'_{p^*p}(0)) \geq 0$ and as $p^* \in C$ it follows from Proposition 4 that p^* satisfies $h(p) \geq h(p^*)$, for all $p \in C$. Thus, from (8) and the equality in (7) we obtain that

$$f(p) \geq h(p) \geq h(p^*) = f(p^*),$$

for all $p \in C$, and the proposition is proved. □

6. Final remarks

A complete, simply connected Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold*, see Examples 4 and 5 above. The Hadamard–Cartan Theorem [17] asserts that the topological and differential structure of a Hadamard manifold coincide with those of a Euclidean space of the same dimension. More precisely, at any point $p \in M$, the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism. Furthermore, for any two points $p, q \in M$ there exists a unique geodesic joining p to q which is minimal. So, Definition 6 becomes

Definition 7. Let $f \in \mathcal{F}$ be a locally Lipschitz function on dom(f). The lower Dini derivative $f' : TM \to [-\infty, +\infty]$ is said to be monotone if

$$f'(p, \exp_p{}^{-1}q) + f'(q, \exp_q{}^{-1}p) \leq 0,$$

for any two points $p, q \in M$.

Since we have obtained results only for locally Lipschitz functions defined on finite dimensional Riemannian manifolds, we expect that the results in this paper will be one more step toward a characterization for Lipschitz and convex functions in more general settings.

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