

Enlargement of Monotone Vector Fields and an Inexact Proximal Point Method for Variational Inequalities in Hadamard Manifolds.

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Abstract

In this paper an inexact proximal point method for variational inequalities in Hadamard manifolds is introduced and studied its convergence properties. The main tool used for presenting the method is the concept of enlargement of monotone vector fields, which generalizes the concept of enlargement of monotone operators from the linear setting to the Riemannian context. As an application, an inexact proximal point method for constrained optimization problems is obtained.

Keywords: Inexact proximal; Hadamard manifolds; enlargement of monotone vector fields; constrained optimization.

1 Introduction

In the last few years, there has been increasing the number of papers dealing with the subject of the extension of concepts and techniques, as well as methods of mathematical programming, from the linear setting to the Riemannian context; papers published in the last three years about this issues include, for example, [1, 3, 4, 5, 7, 8, 9, 10, 12, 26, 31, 35, 41, 42, 43, 44]. Is well known that convexity and monotonicity plays an important role in the analysis and development of methods of mathematical programming. Hence, one of the reasons for this extension is the possibility to transform non-convex or non-monotone problems in Euclidean context into Riemannian convex or monotone problems, by introducing a suitable metric, which allow modify numerical methods to find solutions of these problems; see [10, 11, 19, 21, 36]. These extensions, which in general are nontrivial, are either of purely theoretical nature or aims at obtaining numerical algorithms. Indeed, many mathematical programming problems are naturally posed on Riemannian manifolds having specific underlying geometric and algebraic structure that could be also exploited to reduce the cost of obtaining the solutions; see, e.g., [1, 2, 23, 27, 31, 32, 34, 38, 44].

In this paper, we consider the problem of finding a solution of a variational inequality problem defined on a Riemannian manifold. Variational inequality problems on Riemannian manifolds were first introduced and studied by Németh in [33] for univalued vector fields on Hadamard manifolds and for multivalued vector fields on general Riemannian manifolds by Li and Yao in [29]; for recent works addressing this subject see [24, 30, 39, 40]. It is worth to point out that constrained optimization problems and the problem of finding the zero of a multivalued vector field, studied in [3, 10, 22, 25, 28, 42], are particular instances of the variational inequality problem.

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The aim of this paper is to present an inexact proximal point method for variational inequalities in Hadamard manifolds and to study its convergence properties. As an application, we obtain an inexact proximal point method for constrained optimization problem in Hadamard manifolds. In order to present our method, we first generalize the concept of enlargement of monotone operators, introduced by [16], from linear setting to the Riemannian context; see also [14]. It is worth mentioning that the concept of enlargement of monotone operators in linear spaces has been successfully employed for wide range of purpose; see [15] and its reference therein. As far as we know, this is the first time that the inexact proximal point method for variational inequalities is studied in the Riemannian setting. Finally, we also mention that the method introduced has two important particular instances, namely, the methods (5.1) of [29] and (4.3) of [28].

The organization of the paper is as follows. In Section 1.1, some notations and basic results used in the paper are presented. In Section 2, the concept of enlargement of monotone vector fields is introduced and some properties are obtained. In Section 3 the inexact proximal point method for variational inequalities is presented and studied its convergence properties. As an application, in Section 4 an inexact proximal point method for constrained optimization problems is obtained. Some final remarks are made in Section 5.

1.1 Notation and Terminology

In this section, we introduce some fundamental properties and notations about Riemannian geometry. These basics facts can be found in any introductory book on Riemannian geometry, such as in [17] and [37].

Let M be a n -dimensional Hadamard manifold. *In this paper, all manifolds M are assumed to be Hadamard finite dimensional.* We denote by T_pM the n -dimensional *tangent space* of M at p , by $TM = \cup_{p \in M} T_pM$ *tangent bundle* of M and by $\mathcal{X}(M)$ the space of smooth vector fields on M . The Riemannian metric is denoted by $\langle \cdot, \cdot \rangle$ and the corresponding norm by $\| \cdot \|$. Denote the length of piecewise smooth curves $\gamma : [a, b] \rightarrow M$ joining p to q , i.e., such that $\gamma(a) = p$ and $\gamma(b) = q$, by

$$l(\gamma) = \int_a^b \|\gamma'(t)\| dt,$$

and the Riemannian distance by $d(p, q)$, which induces the original topology on M , namely, (M, d) is a complete metric space and bounded and closed subsets are compact. For $A \subset M$, the notation $\text{int}(A)$ means the interior of the set A , and if A is a nonempty set, the distance from $p \in M$ to A is given by $d(p, A) := \inf\{d(p, q) : q \in A\}$. The metric induces a map $f \mapsto \text{grad } f \in \mathcal{X}(M)$ which associates to each function smooth over M its gradient via the rule $\langle \text{grad } f, X \rangle = df(X)$, $X \in \mathcal{X}(M)$. Let ∇ be the Levi-Civita connection associated to $(M, \langle \cdot, \cdot \rangle)$. A vector field V along γ is said to be *parallel* if $\nabla_{\gamma'} V = 0$. If γ' itself is parallel we say that γ is a *geodesic*. Given that geodesic equation $\nabla_{\gamma'} \gamma' = 0$ is a second order nonlinear ordinary differential equation, then geodesic $\gamma = \gamma_v(\cdot, p)$ is determined by its position p and velocity v at p . It is easy to check that $\|\gamma'\|$ is constant. We say that γ is *normalized* if $\|\gamma'\| = 1$. The restriction of a geodesic to a closed bounded interval is called a *geodesic segment*. Since M is a Hadamard manifolds the length of the geodesic segment γ joining p to q its equals $d(p, q)$, the parallel transport along γ from p to q is denoted by $P_{pq} : T_pM \rightarrow T_qM$. Moreover, *exponential map* $\text{exp}_p : T_pM \rightarrow M$ is defined by $\text{exp}_p v = \gamma_v(1, p)$ is a diffeomorphism and, consequently, M is diffeomorphic to the Euclidean space \mathbb{R}^n , $n = \dim M$. Let $q \in M$ and $\text{exp}_q^{-1} : M \rightarrow T_pM$ be the inverse of the exponential map. Note that $d(q, p) = \|\text{exp}_p^{-1} q\|$, the map $d_q^2 : M \rightarrow \mathbb{R}$ defined by $d_q^2(p) = d(q, p)$ is C^∞ and

$$\text{grad } d_q^2(p) := -2\text{exp}_p^{-1} q. \tag{1}$$

Furthermore, we know that

$$d^2(p_1, p_3) + d^2(p_3, p_2) - 2\langle \exp_{p_3}^{-1} p_1, \exp_{p_3}^{-1} p_2 \rangle \leq d^2(p_1, p_2), \quad p_1, p_2, p_3 \in M. \quad (2)$$

$$\langle \exp_{p_2}^{-1} p_1, \exp_{p_2}^{-1} p_3 \rangle + \langle \exp_{p_3}^{-1} p_1, \exp_{p_3}^{-1} p_2 \rangle \geq d^2(p_2, p_3), \quad p_1, p_2, p_3 \in M. \quad (3)$$

A set, $\Omega \subseteq M$ is said to be *convex* if any geodesic segment with end points in Ω is contained in Ω , that is, if $\gamma : [a, b] \rightarrow M$ is a geodesic such that $x = \gamma(a) \in \Omega$ and $y = \gamma(b) \in \Omega$; then $\gamma((1-t)a + tb) \in \Omega$ for all $t \in [0, 1]$. Given an arbitrary set, $\mathcal{B} \subset M$, the minimal convex subset that contains \mathcal{B} is called the *convex hull* of \mathcal{B} and is denoted by $\text{conv}(\mathcal{B})$; see [19]. Let $\Omega \subset \mathbb{R}^n$ be a convex set, and $p \in \Omega$. Following [28], we define the *normal cone* to Ω at p by

$$N_\Omega(p) := \{w \in T_p M : \langle w, \exp_p^{-1} q \rangle \leq 0, q \in \Omega\}. \quad (4)$$

Let $f : M \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function. The *domain* of f is the set defined by

$$\text{dom} f := \{p \in M : f(p) < \infty\}.$$

The function f is said to be *proper* if $\text{dom} f \neq \emptyset$ and *convex* on a convex set $\Omega \subset \text{dom} f$ if for any geodesic segment $\gamma : [a, b] \rightarrow \Omega$ the composition $f \circ \gamma : [a, b] \rightarrow \mathbb{R}$ is convex. It is very known that d_q^2 is convex. Take $p \in \text{dom} f$. A vector $s \in T_p M$ is said to be a *subgradient* of f at p , if

$$f(q) \geq f(p) + \langle s, \exp_p^{-1} q \rangle, \quad q \in M.$$

The set $\partial f(p)$ of all subgradients of f at p is called the *subdifferential* of f at p . The function f is *lower semicontinuous* at $\bar{p} \in \text{dom} f$ if for each sequence $\{p^k\}$ converging to \bar{p} we have

$$\liminf_{k \rightarrow \infty} f(p^k) \geq f(\bar{p}).$$

Given a multivalued vector field $X : M \rightrightarrows TM$, the domain of X is the set defined by

$$\text{dom} X := \{p \in M : X(p) \neq \emptyset\}, \quad (5)$$

Let $X : M \rightrightarrows TM$ be a vector field and $\Omega \subset M$. We define the following quantity

$$m_X(\Omega) := \sup_{p \in \Omega} \{\|u\| : u \in X(p)\}.$$

We say that X is *locally bounded* if, for all $p \in \text{int}(\text{dom} X)$, there exist an open set $U \subset M$ such that $p \in U$ and there holds $m_X(U) < +\infty$, and *bounded on bounded sets* if for all bounded set $V \subset M$ such that its closure $\bar{V} \subset \text{int}(\text{dom} X)$ it holds that $m_X(V) < +\infty$. The multivalued vector field X is said to be *upper semicontinuous* at $p \in \text{dom} X$ if, for any open set $V \subset T_p M$ such that $X(p) \in V$, there exists an open set $U \subset M$ with $p \in U$ such that $P_{qp} X(q) \subset V$, for any $q \in U$. For two multivalued vector fields X, Y on M , the notation $X \subset Y$ means $X(p) \subset Y(p)$, for all $p \in M$.

A sequence $\{p^k\} \subset (M, d)$ is said to be *quasi-Fejér convergent* to a nonempty set $W \subset M$ if, for every $q \in W$ there exists a summable sequence $\{\epsilon_k\} \subset \mathbb{R}_{++}$, such that $d^2(q, p^{k+1}) \leq d^2(q, p^k) + \epsilon_k$, for $k = 0, 1, \dots$.

We end this section with a result, which its proof is analogous to the proof of Theorem 1 in Burachik et al. [13], by replacing the Euclidean distance by the Riemannian distance.

Proposition 1.1. *Let $\{p^k\}$ be a sequence in (M, d) . If $\{p^k\}$ is quasi-Fejér convergent to non-empty set $W \subset M$, then $\{p^k\}$ is bounded. If furthermore, an accumulation point p of $\{p^k\}$ belongs to W , then $\lim_{k \rightarrow \infty} p^k = p$.*

2 Enlargement of Monotone Vector Fields

A multivalued vector field X is said to be *monotone* if

$$\langle P_{qp}^{-1}u - v, \exp_q^{-1}p \rangle \geq 0, \quad p, q \in \text{dom}X, \quad u \in X(p), \quad v \in X(q), \quad (6)$$

and *strongly monotone*, if there exists $\rho > 0$ such that

$$\langle P_{qp}^{-1}u - v, \exp_q^{-1}p \rangle \geq \rho d^2(p, q), \quad p, q \in \text{dom}X, \quad u \in X(p), \quad v \in X(q). \quad (7)$$

Moreover, a monotone vector field X is said to be *maximal monotone*, if for each $p \in \text{dom}X$ and $u \in T_pM$, there holds:

$$\langle P_{qp}^{-1}u - v, \exp_q^{-1}p \rangle \geq 0, \quad q \in \text{dom}X, \quad v \in X(q) \Rightarrow u \in X(p). \quad (8)$$

Theorem 2.1. *Let f be a proper, lower semicontinuous and convex function on M . The subdifferential ∂f is a monotone multivalued vector field. Furthermore, if $\text{dom}f = M$, then the subdifferential ∂f of f is a maximal monotone vector field.*

Proof. See [28, Theorem 5.1]. □

Lemma 2.1. *Let X_1, X_2 be a maximal monotone vector fields such that $\text{dom}X_1 = \text{dom}X_2 = M$. Then $X_1 + X_2$ is a maximal monotone vector field.*

Proof. Let $z \in M$. Define the following operator $T_1, T_2 : T_zM \rightrightarrows T_zM$ by

$$T_1(u) = P_{\exp_z u, z} X_1(\exp_z u), \quad T_2(u) = P_{\exp_z u, z} X_2(\exp_z u),$$

associated to X_1 and X_2 , respectively. Since the parallel transport is linear, then there holds

$$(T_1 + T_2)(u) = P_{\exp_z u, z} (X_1 + X_2)(\exp_z u), \quad u \in T_zM. \quad (9)$$

Using that X_1 and X_2 are maximal monotone, then it follows from [28, Theorem 3.7] that T_1 and T_2 are upper semicontinuous, $T_1(u)$ and $T_2(u)$ are closed and convex for each $u \in T_zM$. Thus, we conclude that T_1 and T_2 are maximal monotone, see [18, Theorem 2.5, p. 155]. Since T_1 and T_2 are maximal monotone and $\text{dom}(T_1) = \text{dom}(T_2) = T_zM$, we conclude from [6, Corollary 24.4 (i), p. 353] that $T_1 + T_2$ is maximal monotone. Therefore, combining (9) with [28, Theorem 3.7], we conclude that $X_1 + X_2$ is maximal monotone, which conclude the proof. □

Lemma 2.2. *Let X be a maximal monotone vector field such that $\text{dom}X = M$. Then $X + N_\Omega$ is a maximal monotone vector field.*

Proof. The monotonicity of the $X + N_\Omega$ is immediate from the monotonicity of X and definition of N_Ω . Then, take $p \in M$ and let $u \in T_pM$ be such that

$$-\langle u, \exp_p^{-1}q \rangle - \langle v + w, \exp_q^{-1}p \rangle \geq 0, \quad q \in M, \quad v \in X(q), \quad w \in N_\Omega(q). \quad (10)$$

Taking $w = 0$ in last inequality and using the maximality of X we obtain that $u \in X(p)$ and therefore $u + 0 \in (X + N_\Omega)(p)$, which conclude the proof. □

Proposition 2.1. *Let X be a multivalued monotone vector field on M , $q \in M$ and $\lambda > 0$. Then $X + \lambda \text{grad} d_q^2$ is a strongly monotone vector field. Moreover, if X is maximal then $X + \lambda \text{grad} d_q^2$ also maximal.*

Proof. The first part follows by combination of (6), (7) and [20, Proposition 3.2]. The second part follows by straight combination of the convexity of d_q^2 , Theorem 2.1 and Lemma 2.1. \square

Next, we define an operator that play an important rule in this paper.

Definition 2.1. Let X be a multivalued monotone vector field on M and $\epsilon \geq 0$. The enlarged vector field $X^\epsilon : M \rightrightarrows TM$ associated to X is defined by

$$X^\epsilon(p) := \{u \in T_p M : \langle P_{qp}^{-1}u - v, \exp_q^{-1}p \rangle \geq -\epsilon, q \in \text{dom}X, v \in X(q)\}, \quad p \in \text{dom}X. \quad (11)$$

Example 2.1. Let $\epsilon \geq 0$ and $\bar{p} \in M$. Define the closed ball at the origin $0_{T_p M}$ of $T_p M$ and radius $2\sqrt{2\epsilon}$ by

$$B [0_{T_p M}, 2\sqrt{2\epsilon}] := \{w \in T_p M : \|w\| \leq 2\sqrt{2\epsilon}\}.$$

Denote the enlarged vector field of $\partial d_{\bar{p}}^2(p) = \{\text{grad } d_{\bar{p}}^2(p)\}$ by $\partial^\epsilon d_{\bar{p}}^2$. We claim that the following inclusion holds

$$\partial d_{\bar{p}}^2(p) + B [0_{T_p M}, 2\sqrt{2\epsilon}] \subseteq \partial^\epsilon d_{\bar{p}}^2(p), \quad p \in M.$$

Indeed, first note that from (1) we conclude that $\partial d_{\bar{p}}^2(q) = \{-2 \exp_q^{-1} \bar{p}\}$, for each $q \in M$. Due to $\text{dom} \partial d_{\bar{p}}^2 = M$ definition of $\partial^\epsilon d_{\bar{p}}^2$ implies

$$\partial^\epsilon d_{\bar{p}}^2(p) = \{u \in T_p M : -\langle u, \exp_p^{-1}q \rangle + \langle 2 \exp_q^{-1} \bar{p}, \exp_q^{-1}p \rangle \geq -\epsilon, q \in M\}, \quad p \in M. \quad (12)$$

We are going to prove the auxiliary result $\{-2 \exp_p^{-1} \bar{p}\} + A(p) \subset \partial^\epsilon d_{\bar{p}}^2(p)$ for each $p \in M$, where

$$A(p) = \{w \in T_p M : 0 \geq -2d^2(p, q) + \|w\|d(p, q) - \epsilon, q \in M\}, \quad p \in M. \quad (13)$$

First of all, note that by using (3), we obtain the following inequality

$$2 [\langle \exp_p^{-1} \bar{p}, \exp_p^{-1}q \rangle + \langle \exp_q^{-1} \bar{p}, \exp_q^{-1}p \rangle - d^2(p, q)] \geq 0, \quad p, q \in M.$$

Take $w \in A(p)$. Since $\langle w, \exp_p^{-1}q \rangle \leq \|w\|d(p, q)$, for all $w \in A(p)$ and $p, q \in M$, combining (13) with last inequality yields

$$2 [\langle \exp_p^{-1} \bar{p}, \exp_p^{-1}q \rangle + \langle \exp_q^{-1} \bar{p}, \exp_q^{-1}p \rangle - d^2(p, q)] \geq -2d^2(p, q) + \langle w, \exp_p^{-1}q \rangle - \epsilon, \quad p, q \in M.$$

Simple algebraic manipulations in last inequality shows that it is equivalent to the following ones

$$-\langle -2 \exp_p^{-1} \bar{p} + w, \exp_p^{-1}q \rangle + \langle 2 \exp_q^{-1} \bar{p}, \exp_q^{-1}p \rangle \geq -\epsilon, \quad p, q \in M,$$

which, from (12), allows to conclude that $-2 \exp_p^{-1} \bar{p} + w \in \partial^\epsilon d_{\bar{p}}^2(p)$, for all $w \in A(p)$ and $p \in M$. Thus, the auxiliary result is proved. Finally, note that $w \in A(p)$ if, and only if, there holds $\|w\|^2 - 8\epsilon < 0$, or equivalently, $\|w\| < 2\sqrt{2\epsilon}$. Therefore, $A(p) = B [0_{T_p M}, 2\sqrt{2\epsilon}]$ and, because $\partial d_{\bar{p}}^2(p) + A(p) \subset \partial^\epsilon d_{\bar{p}}^2(p)$ for each $p \in M$, the proof of the claim is done.

Remark 2.1. Note that if M has zero curvature then the inequality (3) holds as a equality. Therefore, in Example 2.1, we can prove that the inequality holds as equality, namely,

$$\partial d_{\bar{p}}^2(p) + B [0_{T_p M}, 2\sqrt{2\epsilon}] = \partial^\epsilon d_{\bar{p}}^2(p), \quad p \in M.$$

Proposition 2.2. Let X be a monotone vector field on M and $\epsilon \geq 0$. Then, $X \subset X^\epsilon$ and $\text{dom}X \subset \text{dom}X^\epsilon$. In particular, if $\text{dom}X = M$ then $\text{dom}X^\epsilon = \text{dom}X$. Moreover, if X is maximal then $X^0 = X$.

Proof. Take $\epsilon \geq 0$. Since X is monotone, the first part of the proposition follows straightly from (6) and (11). Thus, using that $\text{dom}X = M$, we conclude that $\text{dom}X^\epsilon = \text{dom}X$. The proof of the last part, follows by combining the definition in (11) and maximality of X , and by taking into account that $X \subset X^0$. \square

Proposition 2.3. *Let X, X_1 and X_2 be multivalued monotone vector fields on M and $\epsilon, \epsilon_1, \epsilon_2 \geq 0$. Then, there hold:*

- i) *If $\epsilon_1 \geq \epsilon_2 \geq 0$ then $X^{\epsilon_2} \subset X^{\epsilon_1}$;*
- ii) *$X_1^{\epsilon_1} + X_2^{\epsilon_2} \subset (X_1 + X_2)^{\epsilon_1 + \epsilon_2}$;*
- iii) *$X^\epsilon(p)$ is closed and convex for all $p \in M$;*
- iv) *$\alpha X^\epsilon = (\alpha X)^{\alpha\epsilon}$ for all $\alpha \geq 0$;*
- v) *$\alpha X_1^\epsilon + (1 - \alpha)X_2^\epsilon \subset (\alpha X_1 + (1 - \alpha)X_2)^\epsilon$ for all $\alpha \in [0, 1]$;*
- vi) *If $E \subset \mathbb{R}_+$, then $\bigcap_{\epsilon \in E} X^\epsilon = X^{\bar{\epsilon}}$ with $\bar{\epsilon} = \inf E$.*

Proof. The proof is a consequence of Definition 2.1 by using simple algebraic manipulations. \square

Proposition 2.4. *Let X be a multivalued monotone vector fields on M , $\{\epsilon^k\}$ be a sequence of positive numbers and $\{(p^k, u^k)\}$ a sequence in TM . If $\bar{\epsilon} = \lim_{k \rightarrow \infty} \epsilon^k$, $\bar{p} = \lim_{k \rightarrow \infty} p^k$, $\bar{u} = \lim_{k \rightarrow \infty} u^k$ and $u^k \in X^{\epsilon^k}(p^k)$ for all k , then $\bar{u} \in X^{\bar{\epsilon}}(\bar{p})$;*

Proof. Since $u^k \in X^{\epsilon^k}(p^k)$ for all k , then from Definition 2.1 we have

$$-\langle u^k, \exp_{p^k}^{-1} q \rangle + \langle -v, \exp_q^{-1} p^k \rangle \geq -\epsilon_k, \quad q \in \text{dom}X, \quad v \in X(q).$$

Taking the limit in the last inequality, as k goes to ∞ , we obtain

$$-\langle \bar{u}, \exp_{\bar{p}}^{-1} q \rangle + \langle -v, \exp_q^{-1} \bar{p} \rangle \geq -\bar{\epsilon}, \quad q \in \text{dom}X, \quad v \in X(q).$$

Therefore, using again Definition 2.1 the result follows. \square

Proposition 2.5. *Suppose that X is maximal monotone and $\text{dom}X = M$. Then X is locally bounded on M .*

Proof. See [28, Lemma 3.6]. \square

Proposition 2.6. *If X is maximal monotone and $\text{dom}X = M$ then X^ϵ is bounded on bounded sets, for all $\epsilon \geq 0$.*

Proof. Since X is monotone and $\text{dom}X = M$, Proposition 2.2 implies that $\text{dom}X^\epsilon = M$. Take $V \subset M = \text{int}(\text{dom}X^\epsilon)$ a bounded set. Note that $\bar{V} \subset \text{int}(\text{dom}X^\epsilon)$. Let $r > 0$ and define the set $V_r = \{p \in M : d(p, V) \leq r\}$. Taking into account that $\text{dom}X = M$, then $V_r \subset \text{dom}X$. Moreover, since both sets V and V_r are bounded, Proposition 2.5 implies that $m_X(V) < +\infty$ and $m_X(V_r) < +\infty$. We are going to prove that

$$m_{X^\epsilon}(V) \leq \frac{\epsilon}{r} + m_X(V_r) + 2m_X(V). \quad (14)$$

Take $p \in V$, $u \in X^\epsilon(p)$. Thus, for all $v \in X(q)$, the definition of $X^\epsilon(p)$ in (11) implies

$$-\epsilon \leq -\langle u, \exp_p^{-1} q \rangle - \langle v, \exp_q^{-1} p \rangle.$$

Let $\hat{u} \in X(p)$. For $\hat{u} \neq u$ define $q = \exp_p w$, where $w = (r/\|u - \hat{u}\|)(u - \hat{u})$. Thus, last inequality becomes

$$-\epsilon \leq -\|u - \hat{u}\|r - \langle \hat{u}, \exp_p^{-1} q \rangle - \langle v, \exp_q^{-1} p \rangle.$$

Using that the parallel transport is an isometry, we conclude from last inequality that

$$-\epsilon \leq -\|u - \hat{u}\|r + \|\exp_q^{-1} p\| \|P_{qp}^{-1} \hat{u} - v\|.$$

Since $r = \|\exp_q^{-1} p\|$, using triangle inequality and once again that the parallel transport is an isometry, some manipulation in last inequality yields $\|u - \hat{u}\| \leq \epsilon/r + \|\hat{u}\| + \|v\|$. Hence, taking into account that $\|u\| \leq \|u - \hat{u}\| + \|\hat{u}\|$, we obtain

$$\|u\| \leq \frac{\epsilon}{r} + 2\|\hat{u}\| + \|v\|.$$

Note that last inequality also holds for $u = \hat{u}$. Since $\|\exp_q^{-1} p\| = r$ and $p \in V$, we have $q \in V_r$. Thus, $\|\hat{u}\| \leq m_X(\Omega)$ and $\|v\| \leq m_X(\Omega_r)$, which implies that

$$\|u\| \leq \frac{\epsilon}{r} + m_X(\Omega_r) + 2m_X(\Omega).$$

Since u is an arbitrary element of $X^\epsilon(\Omega)$, the inequality in (14) follows, and the proof is concluded. \square

3 An Inexact Proximal Point Method for Variational Inequalities

Let $X : M \rightrightarrows TM$ be a multivalued vector field and $\Omega \subset M$ be a nonempty set. The *variational inequality problem* $\text{VIP}(X, \Omega)$ consists of finding $p^* \in \Omega$ such that there exists $u \in X(p^*)$ satisfying

$$\langle u, \exp_{p^*}^{-1} q \rangle \geq 0, \quad q \in \Omega.$$

Using (4), i.e., the definition of normal cone to Ω , the $\text{VIP}(X, \Omega)$ becomes the problem of finding $p^* \in \Omega$ satisfying the inclusion

$$0 \in X(p) + N_\Omega(p). \quad (15)$$

Remark 3.1. *In particular, if $\Omega = M$, then $N_\Omega(p) = \{0\}$ and $\text{VIP}(X, \Omega)$ becomes to the problem of finding $p^* \in \Omega$ such that $0 \in X(p^*)$.*

From now on $S(Y, \Omega)$ denotes the solution set of the inclusion (15). We need of the following three assumptions:

A1. $Y := X + N_\Omega$ with $\text{dom} X = M$ and Ω closed and convex;

A2. X is maximal monotone;

A3. $S(X, \Omega) \neq \emptyset$.

Take $0 < \hat{\lambda} \leq \tilde{\lambda}$, a sequence $\{\lambda_k\} \subset \mathbb{R}$ such that $\hat{\lambda} \leq \lambda_k \leq \tilde{\lambda}$ and a sequence $\{\epsilon_k\} \subset \mathbb{R}_{++}$ such that $\sum_{k=0}^{\infty} \epsilon_k < \infty$. The *proximal point method* for $\text{VIP}(X, \Omega)$ is defined as follows: Given $p^0 \in \Omega$ take p^{k+1} such that

$$0 \in (X^{\epsilon_k} + N_\Omega)(p^{k+1}) - 2\lambda_k \exp_{p^{k+1}}^{-1} p^k, \quad k = 0, 1, \dots \quad (16)$$

Remark 3.2. The method (16) has many important particular instances. For example, in the case $\epsilon_k = 0$ for all k , we obtain the method (5.1) of [29]. For $\Omega = M$ and $\epsilon_k = 0$ for all k , we obtain the method (4.3) of [28]. For $M = \mathbb{R}^n$, we obtain the method (23)-(25) of [16], where the Bregman distance is induced by the square of the Euclidean norm and $C = \mathbb{R}^n$.

Lemma 3.1. For each $q \in M$ and $\lambda > 0$ the following inclusion problem

$$0 \in X(p) - 2\lambda \exp_p^{-1} q + N_\Omega(p), \quad p \in M.$$

has an unique solution.

Proof. Since X is a monotone vector field and $\lambda > 0$, combining Proposition 2.1 with (1), we conclude that the vector field $Z(p) = X(p) - 2\lambda \exp_p^{-1} q$ is a strongly maximal monotone vector field. Therefore, using that Z is maximal and taking into account that M is a Hadamard manifold and Ω is a nonempty and convex set, we may combine [28, Proposition 3.5] with [29, Corollary 3.14] to conclude the proof. \square

Now we are going to prove the convergence result for the proximal point method (16).

Theorem 3.1. Assume that **A1-A3** hold. Then, the sequence $\{p^k\}$ generated by (16) is well defined and converges to a point $p^* \in S(X, \Omega)$.

Proof. Since $\text{dom}X = M$, Proposition 2.2 and item i of Proposition 2.6 imply that $X(p) \subseteq X^{\epsilon_k}(p)$ for all $p \in M$ and $k = 0, 1, \dots$. Hence, for proving the well definition of the sequence $\{p^k\}$ it is sufficient to prove that the inclusion

$$0 \in X(p) - 2\lambda_k \exp_p^{-1} p^k + N_\Omega(p), \quad p \in M,$$

has solution, for each $k = 0, 1, \dots$, which is a consequence of Lemma 3.1.

Now, we are going to prove the convergence of $\{p^k\}$ to a point $p^* \in S(X, \Omega)$. Using Proposition 2.2 we conclude that $N_\Omega \subset N_\Omega^0$. Thus, from item ii of Proposition 2.3 we have $X^{\epsilon_k} + N_\Omega \subset (X + N_\Omega)^{\epsilon_k}$, for all $k = 0, 1, \dots$. Therefore, using (16) we obtain

$$2\lambda_k \exp_{p^{k+1}}^{-1} p^k \in (X + N_\Omega)^{\epsilon_k}(p^{k+1}), \quad k = 0, 1, \dots \quad (17)$$

Since $P_{qp^{k+1}}^{-1} \exp_q^{-1} p^{k+1} = -\exp_{p^{k+1}}^{-1} q$ and the parallel transport is a isometry, last inclusion together with Definition 2.1 yield

$$-2\lambda_k \left\langle \exp_{p^{k+1}}^{-1} p^k, \exp_{p^{k+1}}^{-1} q \right\rangle + \left\langle v, -\exp_q^{-1} p^{k+1} \right\rangle \geq -\epsilon_k, \quad q \in \Omega, \quad v \in (X + N_\Omega)(q), \quad k = 0, 1, \dots$$

Particularly, if $q \in S(X, \Omega)$ then $0 \in X + N_\Omega(q)$ and last inequality becomes

$$-2\lambda_k \left\langle \exp_{p^{k+1}}^{-1} p^k, \exp_{p^{k+1}}^{-1} q \right\rangle \geq -\epsilon_k, \quad q \in S(X, \Omega), \quad k = 0, 1, \dots$$

Using last inequality and (2) with $p_1 = p^k$, $p_2 = q$ and $p_3 = p^{k+1}$, after some algebras we obtain

$$-\frac{\epsilon_k}{2\lambda_k} \leq d^2(q, p^k) - d^2(p^k, p^{k+1}) - d^2(q, p^{k+1}), \quad q \in S(X, \Omega), \quad k = 0, 1, \dots \quad (18)$$

Since $0 < \hat{\lambda} \leq \lambda_k$, the last inequality gives

$$d^2(q, p^{k+1}) \leq d^2(q, p^k) + \frac{\epsilon_k}{\hat{\lambda}}, \quad q \in S(X, \Omega), \quad k = 0, 1, \dots \quad (19)$$

Because $\sum_{k=0}^{\infty} \epsilon_k < \infty$ and $S(X, \Omega) \neq \emptyset$, last inequality implies that $\{p^k\}$ is quasi-Fejér convergent to $S(X, \Omega)$. From Proposition 1.1, for concluding the proof is sufficient to prove that there exists an accumulation point \bar{p} of $\{p^k\}$ belonging to $S(X, \Omega)$. Since $\{p^k\}$ is quasi-Fejér convergent to $S(X, \Omega)$, Proposition 1.1 implies that $\{p^k\}$ is bounded. Take \bar{p} and $\{p^{n_k}\}$ an accumulation point and a subsequence of $\{p^k\}$, respectively, such that $\bar{p} = \lim_{k \rightarrow \infty} p^{n_k}$. On the other hand, since $0 < \hat{\lambda} \leq \lambda_k$ and $\sum_{k=0}^{\infty} \epsilon_k < \infty$, the inequality in (18) implies that $\lim_{k \rightarrow \infty} d(p^k, p^{k+1}) = 0$. Thus, $\lim_{k \rightarrow \infty} \exp_{p^{n_k+1}}^{-1} p^{n_k} = 0$ and $\lim_{k \rightarrow \infty} p^{n_k+1} = \bar{p}$. Now, using (17) we have

$$2\lambda_{n_k} \exp_{p^{n_k+1}}^{-1} p^{n_k} \in (X + N_{\Omega})^{\epsilon_{n_k}}(p^{n_k+1}), \quad k = 0, 1, \dots$$

Therefore, letting k goes to ∞ in the last inclusion and using Proposition 2.4, Lemma 2.2, Proposition 2.2 and taking into account that $\{\lambda_k\}$ is bounded we obtain

$$0 \in (X + N_{\Omega})(\bar{p}),$$

which implies that $\bar{p} \in S(X, \Omega)$ and the proof is concluded. \square

4 An Inexact Proximal Point Method for Otimization

Throughout this section, we assume that $f : M \rightarrow \mathbb{R}$ is a convex function. The enlargement of the subdifferential of f , denoted by $\partial^{\epsilon} f : M \rightrightarrows TM$, is defined by

$$\partial^{\epsilon} f(p) := \{u \in T_p M : \langle P_{qp}^{-1} u - v, \exp_q^{-1} p \rangle \geq -\epsilon, q \in M, v \in \partial f(q)\}, \quad \epsilon \geq 0.$$

and we denote the ϵ -subdifferential of f by $\partial_{\epsilon} f : M \rightrightarrows TM$, which is given by

$$\partial_{\epsilon} f(p) := \{u \in T_p M : f(q) \geq f(p) + \langle u, \exp_p^{-1} q \rangle - \epsilon, q \in M\}, \quad \epsilon \geq 0.$$

Example 4.1. Let $\epsilon \geq 0$ and $\bar{p} \in M$. Define the closed ball at the origin $0_{T_p M}$ of $T_p M$ and radius $2\sqrt{\epsilon}$ by

$$B [0_{T_p M}, 2\sqrt{\epsilon}] := \{w \in T_p M : \|w\| \leq 2\sqrt{\epsilon}\}.$$

Denote the ϵ -subdifferential of $\partial d_{\bar{p}}^2(p) = \{\text{grad } d_{\bar{p}}^2(p)\}$ by $\partial_{\epsilon} d_{\bar{p}}^2$. We claim that the following inclusion holds

$$\partial d_{\bar{p}}^2(p) + B [0_{T_p M}, 2\sqrt{\epsilon}] \subseteq \partial_{\epsilon} d_{\bar{p}}^2(p), \quad p \in M.$$

Indeed, first note that from (1) we conclude that $\partial d_{\bar{p}}^2(q) = \{-2 \exp_q^{-1} \bar{p}\}$, for each $q \in M$. Due to $\text{dom} \partial d_{\bar{p}}^2 = M$ definition of $\partial_{\epsilon} d_{\bar{p}}^2$ implies

$$\partial_{\epsilon} d_{\bar{p}}^2(p) = \{u \in T_p M : d^2(\bar{p}, q) \geq d^2(\bar{p}, p) + \langle u, \exp_p^{-1} q \rangle - \epsilon, q \in M\}, \quad p \in M. \quad (20)$$

We are going to prove the auxiliary result $\{-2 \exp_p^{-1} \bar{p}\} + A(p) \subset \partial_{\epsilon} d_{\bar{p}}^2(p)$ for each $p \in M$, where

$$B(p) = \{w \in T_p M : 0 \geq -d^2(p, q) + \|w\|d(p, q) - \epsilon, q \in M\}, \quad p \in M. \quad (21)$$

First of all, note that by using (2), we obtain the following inequality

$$d^2(\bar{p}, q) - d^2(\bar{p}, p) - d^2(p, q) + 2\langle \exp_p^{-1} \bar{p}, \exp_p^{-1} q \rangle \geq 0, \quad p, q \in M.$$

Take $w \in B(p)$. Since $\langle w, \exp_p^{-1} q \rangle \leq \|w\|d(p, q)$, for all $w \in B(p)$ and $p, q \in M$, combining (21) with last inequality yields

$$d^2(\bar{p}, q) - d^2(\bar{p}, p) - d^2(p, q) + 2\langle \exp_p^{-1} \bar{p}, \exp_p^{-1} q \rangle \geq -d^2(p, q) + \langle w, \exp_p^{-1} q \rangle - \epsilon, \quad p, q \in M.$$

Simple algebraic manipulations in last inequality shows that it is equivalent to the following ones

$$d^2(\bar{p}, q) \geq d^2(\bar{p}, p) + \langle -2 \exp_p^{-1} \bar{p} + w, \exp_p^{-1} q \rangle - \epsilon, \quad p, q \in M,$$

which, from (20), allows to conclude that $-2 \exp_p^{-1} \bar{p} + w \in \partial_\epsilon d_p^2(p)$, for all $w \in B(p)$ and $p \in M$. Thus, the auxiliary result is proved. Finally, note that $w \in B(p)$ if, and only if, there holds $\|w\|^2 - 4\epsilon < 0$, or equivalently, $\|w\| < 2\sqrt{\epsilon}$. Therefore, $B(p) = B[0_{T_p M}, 2\sqrt{\epsilon}]$ and, because $\partial d_p^2(p) + A(p) \subset \partial_\epsilon d_p^2(p)$ for each $p \in M$, the proof of the claim is done.

Proposition 4.1. For each $p \in M$, there holds $\partial_\epsilon f(p) \subseteq \partial^\epsilon f(p)$.

Proof. Take $u \in \partial_\epsilon f(p)$, $q \in M$ and $v \in \partial f(q)$. From the definitions of $\partial f(q)$ and $\partial_\epsilon f(p)$ we have

$$f(p) \geq f(q) + \langle v, \exp_q^{-1} p \rangle, \quad f(q) \geq f(p) + \langle u, \exp_p^{-1} q \rangle - \epsilon,$$

respectively. Combining two last inequalities we conclude that $0 \geq \langle v, \exp_q^{-1} p \rangle + \langle u, \exp_p^{-1} q \rangle + \epsilon$. Since the parallel transport is an isometry and $P_{qp}^{-1} \exp_p^{-1} q = -\exp_q^{-1} p$, last inequality becomes

$$0 \geq \langle v, \exp_q^{-1} p \rangle + \langle P_{qp}^{-1} u, -\exp_q^{-1} p \rangle - \epsilon.$$

Thus, using last inequality and definition of $\partial^\epsilon f(p)$ we obtain that $u \in \partial^\epsilon f(p)$. Therefore, the prove is done. \square

Remark 4.1. Note that if M has zero curvature then the inequality (2) holds as a equality. Therefore, in Example 4.1, we can prove that the equality holds as equality, namely,

$$\partial d_p^2(p) + B[0_{T_p M}, 2\sqrt{\epsilon}] = \partial_\epsilon d_p^2(p), \quad p \in M.$$

Moreover, we can also prove that the inclusion $\partial_\epsilon d_p^2(p) \subset \partial^\epsilon d_p^2(p)$ is strict, for all $p \in M$, see Example 2.1.

Let $\Omega \subset M$. The constrained optimization problem consists in

$$\text{Minimize } f(p), \quad \text{subject to } p \in \Omega. \quad (22)$$

Letting δ_Ω be the indicate function, defined by $\delta_\Omega(p) = 0$, if $p \in \Omega$ and $\delta_\Omega(p) = +\infty$ otherwise, Problem 22 is equivalent to

$$\text{Minimize } (f + \delta_\Omega)(p), \quad \text{subject to } p \in M.$$

From now on, $\Omega \subset M$ is a closed and convex set and $S(f, \Omega)$ denotes the solution set of Problem 22.

Theorem 4.1. There holds $\partial(f + \delta_\Omega)(p) = \partial f(p) + N_\Omega(p)$, for each $p \in \Omega$. Moreover, $p^* \in S(f, \Omega)$ if, and only if, $0 \in \partial f(p^*) + N_\Omega(p^*)$.

Proof. The first part was proved in [28, Proposition 5.4]. To prove the second part, first use convexity of Ω and f for concluding that $f + \delta_\Omega$ is also convex, and then use the first part to obtain the result. \square

Take $0 < \hat{\lambda} \leq \tilde{\lambda}$, a sequence $\{\lambda_k\} \subset \mathbb{R}$ such that $\hat{\lambda} \leq \lambda_k \leq \tilde{\lambda}$ and a sequence $\{\epsilon_k\} \subset \mathbb{R}_{++}$ such that $\sum_{k=0}^{\infty} \epsilon_k < \infty$. The inexact proximal point method for the constrained optimization problem in (22) is defined as follows:

Inicialization:

$$p^0 \in \Omega. \quad (23)$$

Iterative Step: Given p^k , define $X_k : M \rightrightarrows TM$ as

$$X_k(p) := (\partial^{\epsilon_k} f + N_\Omega)(p) - 2\lambda_k \exp_p^{-1} x^k, \quad (24)$$

and take p^{k+1} such that

$$0 \in X_k(p^{k+1}). \quad (25)$$

Remark 4.2. For $\epsilon_k = 0$ the above method generalizes the method (5.15) of Chong Li et. al. [28] and, for $\epsilon_k = 0$ and $\Omega = M$ we obtain the method proposed by Ferreira and Oliveira [25].

Theorem 4.2. Assume that $S(f, \Omega) \neq \emptyset$. Then, the sequence $\{p^k\}$ generated by (23)-(25) is well defined and converges to a point $p^* \in S(f, \Omega)$.

Proof. Since $\text{dom} f = M$, Theorem 2.1 implies that ∂f is maximal monotone. Therefore, taking into account that $N_\Omega = \partial\delta_\Omega$, the result follows directly from Theorem 3.1 with $X = \partial f$. \square

5 Final Remarks

In this paper we study some basics properties of enlargement of monotone vector fields. Since this concept has been successfully employed for wide range of purpose, in linear setting, we expect that the results of this paper become a first step towards a more general theory in the Riemannian context, including other algorithms for solving variational inequalities. We foresee further progress in this topic in the nearby future.

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