On the spherical convexity of quadratic functions *

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Abstract

In this paper we study the spherical convexity of quadratic functions on spherically convex sets. In particular, conditions characterizing the spherical convexity of quadratic functions on spherical convex sets associated to the positive orthants, Lorentz and circular cones are given.

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1 Introduction

In this paper we study the spherical convexity of quadratic functions on spherical convex sets. This problem arised by trying to make certain fixed point theorems, surjectivity theorems, and existence theorems for complementarity problems and variational inequalities more explicit. Under some smoothness conditions, these existence theorems could be reduced to optimizing a quadratic function on the intersection of the sphere and a cone. Such intersections are exactly the spherically convex sets [5]. Hence, the optimization problem reduces to optimizing quadratic functions on spherically convex sets. When these quadratic functions are spherically convex [6] the spherical local minimum is equal to the spherical global minimum. Therefore, it is natural to consider the problem of determining the spherically convex quadratic functions on spherically convex sets. The optimization problem above is in fact the problem of calculating the scalar derivative, introduced by S. Z. Németh in [18–20], along cones [22]. More precisely, under certain smoothness conditions, in Euclidean spaces and for pointed closed convex cones, [9, Theorem 7.2, Corollary 7.2, Theorem 7.3, Corollary 7.2, Theorem 7.4, Corollary 7.3, Theorem 7.11], [10, Theorem 6.3, Theorem 6.4], [12, Corollary 8.1], [11, Corollary 9.1] need the calculation of the scalar derivative which can be achieved by solving the optimization problems in [22, Theorem 17 or Theorem 18] or [12, Theorem 4.6], which are problems of optimizing quadratic functions on spherically convex sets. In fact, by using [22, Theorem 15 or Theorem 16] or [12, Theorem 4.1], under certain smoothness conditions, in Euclidean spaces and for pointed closed convex cones, [13, Corollary 8.1] can also be reduced to optimizing quadratic functions on spherically convex sets. The corresponding theorems and corollaries above can also be found in [14].

Apart from the motivation of solving fixed point theorems, surjectivity theorems, and existence theorems for complementarity problems and variational inequalities by calculating the scalar derivative, the motivation of this study is much wider. It has both theoretical and applied nature, since

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it aims at obtaining efficient algorithms (see, e.g., [2,4,7,27,28,30–33]) for constrained optimization problems on the sphere. Indeed, many optimization problems are naturally posed on the sphere, which has a specific underlining algebraic structure that could be exploited to greatly reduce the cost of obtaining the solutions; see [27, 28, 32, 33]. Besides the theoretical interest, constrained optimization problems on the sphere also have a wide range of applications in many different areas of study such as numerical multilinear algebra (see, e.g., [23]), solid mechanics (see, e.g., [8]), signal processing (see, e.g., [24, 29]), computational anatomy (see, e.g., [7]) and quantum mechanics (see, e.g., [1]). For instance, consider the generic constrained optimization problem on the sphere:

$$\min\{f(x) : x \in C\}, \qquad C \subseteq \mathbb{S}^n. \tag{1}$$

For $C = \mathbb{S}^n$ and a quadratic form $f(x) = x^T Qx$, the problem in (1) becomes a minimal eigenvalue problem, that is, finding the spectral norm of the matrix $-Q$ (see, e.g., [27]). Problem (1) includes as particular cases the problem of deciding the non-negativity of a homogeneous multivariate polynomial over the sphere (see, e.g., [16,25,26]) as well as the Bi-Quadratic Optimization problem over unit spheres (see, e.g., [17]). For quadratic functions it also contains the trust region problem that appears in many nonlinear programming algorithms as a sub-problem, see [3].

The goal of the paper is to present conditions satisfied by quadratic functions which are spherically convex on spherical convex sets. Besides, we present conditions characterizing the spherical convexity of quadratic functions on spherically convex sets associated to the circular cones and the positive orthant cone.

The remainder of this paper is organized as follows. In Section 2, we recall some notations and basic results used throughout the paper. In Section 3 we present some general properties satisfied by quadratic functions which are spherically convex. In Section 4 we present a condition characterizing the spherical convexity of quadratic functions on the spherical convex set defined by the positive orthant cone. In Section 5 we present a condition characterizing the spherical convexity of quadratic functions on spherical convex sets defined by circular cones. We conclude this paper by making some final remarks in Section 6.

2 Notations and basic results

In this section we present the notations and some auxiliary results used throughout the paper. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the canonical inner product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$. Denote by $\mathbb{R}^n_+$ the nonnegative orthant and by $\mathbb{R}^n_{++}$ the positive orthant. The notation $x \perp y$ means that $\langle x, y \rangle = 0$. Denote by e^i the i-th canonical unit vector in $\mathbb{R}^n$. The *unit sphere* is denoted by

$$\mathbb{S} := \{x \in \mathbb{R}^n \ : \ \|x\| = 1\}.$$

The *dual cone* of a cone $\mathcal{K} \subset \mathbb{R}^n$ is the cone $\mathcal{K}^* := \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0, \ \forall\, y \in \mathcal{K}\}$. Any pointed closed convex cone with nonempty interior will be called *proper cone*. $\mathcal{K}$ is called *subdual* if $\mathcal{K} \subset \mathcal{K}^*$, *superdual* if $\mathcal{K}^* \subset \mathcal{K}$ and *self-dual* if $\mathcal{K}^* = \mathcal{K}$. $\mathcal{K}$ is called *strongly superdual* if $\mathcal{K}^* \subset \operatorname{int}(\mathcal{K})$. The set of all $m \times n$ matrices with real entries is denoted by $\mathbb{R}^{m\times n}$ and $\mathbb{R}^n \equiv \mathbb{R}^{n\times 1}$. In Section 5 we will also use the identification $\mathbb{R}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R}$, which makes the notations much easier. The matrix $\mathrm{I_n}$ denotes the $n \times n$ identity matrix. If $x \in \mathbb{R}^n$ then $\operatorname{diag}(x)$ will denote an $n \times n$ diagonal matrix with (i, i)-th entry equal to x_i, for $i = 1, \ldots, n$. For $a \in \mathbb{R}$ and $B \in \mathbb{R}^{(n-1)\times(n-1)}$ we denote $\operatorname{diag}(a, B) \in \mathbb{R}^{n\times n}$ the matrix defined by

$$\operatorname{diag}(a, B) := \begin{bmatrix} a & 0 \\ 0 & B \end{bmatrix}.$$

Recall that a *Z-matrix* is a matrix with nonpositive off-diagonal elements. Let $\mathcal{K} \subset \mathbb{R}^n$ be a pointed closed convex cone with nonempty interior, the *$\mathcal{K}$-Z-property* of a matrix $A \in \mathbb{R}^{n\times n}$ means that $\langle Ax, y\rangle \leq 0$, for any $(x, y) \in C(\mathcal{K})$, where $C(\mathcal{K}) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \in \mathcal{K},\ y \in \mathcal{K}^*,\ x \perp y\}$. The matrix $A \in \mathbb{R}^{n\times n}$ is said to have the *$\mathcal{K}$-Lyapunov-like property* if A and $-A$ have the $\mathcal{K}$-Z-property, and is said to be *$\mathcal{K}$-copositive* if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{K}$. If $\mathcal{K} = \mathbb{R}^n_+$, then the $\mathcal{K}$-Z-property of a matrix coincides with the matrix being a Z-matrix and the $\mathcal{K}$-Lyapunov-like property with the matrix being diagonal.

The intersection curve of a plane though the origin of $\mathbb{R}^n$ with the sphere $\mathbb{S}$ is called a *geodesic*. A geodesic segment $\gamma : [a, b] \to \mathbb{S}^n$ is said to be *minimal* if its arc length is equal to the intrinsic distance between its end points, i.e., if $\ell(\gamma) := \arccos\langle\gamma(a), \gamma(b)\rangle$. The set $C \subseteq \mathbb{S}$ is said to be *spherically convex* if for any x, $y \in C$ all the minimal geodesic segments joining x to y are contained in C. Let $C \subset \mathbb{S}$ be a spherically convex set and $I \subset \mathbb{R}$ an interval. The following result is proved in [5].

Proposition 1. *Let $K_C := \{tp : p \in C,\ t \in [0, +\infty)\}$ be the cone generated by the set $C \subset \mathbb{S}^n$. The set C is spherically convex if and only if the associated cone K_C is convex and pointed.*

A function $f : C \to \mathbb{R}$ is said to be *spherically convex (respectively, strictly spherically convex)* if for any minimal geodesic segment $\gamma : I \to C$, the composition $f \circ \gamma : I \to \mathbb{R}$ is convex (respectively, strictly convex) in the usual sense. The next result is an immediate consequence of [6, Propositions 8 and 9].

Proposition 2. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a proper cone, $\mathcal{C} = \mathrm{int}(\mathcal{K}) \cap \mathbb{S}$ and $f : \mathcal{C} \to \mathbb{R}$ a differentiable function. Then, the following statements are equivalent:*

(i) f is spherically convex;

(ii) $\langle Df(x) - Df(y), x - y\rangle + (\langle x, y\rangle - 1)\left[\langle Df(x), x\rangle + \langle Df(y), y\rangle\right] \geq 0$, for all $x, y \in \mathcal{C}$;

(iii) $\left\langle D^2 f(y)x, x\right\rangle - \langle Df(y), y\rangle \geq 0$, for all $x, y \in \mathcal{C}$ with $x \perp y$.

It is well known that if $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then Q defines a linear orthogonal mapping, which is an isometry of the sphere. In the following remark we state some important properties of the isometries of the sphere, for that, given $\mathcal{C} \subset \mathbb{S}$ and $Q \in \mathbb{R}^{n\times n}$, we define

$$Q\mathcal{C} := \{Qx \ : \ x \in \mathcal{C}\}.$$

Remark 1. *Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix, i.e., $Q^T = Q^{-1}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ be spherically convex sets. Then $\tilde{\mathcal{C}}_2 := Q\mathcal{C}_2$ is a spherically convex set. Hence, if $\tilde{\mathcal{C}}_2 \subset \tilde{\mathcal{C}}_1$ and $f : \tilde{\mathcal{C}}_1 \to \mathbb{R}$ is a spherically convex function, then $h := f \circ Q : \mathcal{C}_2 \to \mathbb{R}$ is also a spherically convex function. In particular, if $\tilde{\mathcal{C}}_2 = \tilde{\mathcal{C}}_1$ then, $f : \tilde{\mathcal{C}}_1 \to \mathbb{R}$ is spherically convex if, only if, $h := f \circ Q : \mathcal{C}_2 \to \mathbb{R}$ is spherically convex.*

We will show next a useful property of proper cones which will be used in the Section 5.

Lemma 1. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a proper cone. If $x \in \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ such that $x \perp y$, then $x \notin \mathrm{int}(\mathcal{K}^*) \cup -\mathrm{int}(\mathcal{K}^*)$.*

Proof. If $x \in \mathrm{int}(\mathcal{K}^*)$, then $\langle x, y\rangle > 0$ and if $x \in -\mathrm{int}(\mathcal{K}^*)$, then $\langle x, y\rangle < 0$. Hence, $x \in \mathbb{S}$, $y \in \mathcal{K} \cap \mathbb{S}$ and $x \perp y$ imply $x \notin \mathrm{int}(\mathcal{K}^*) \cup -\mathrm{int}(\mathcal{K}^*)$. □

Let $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. For a quadratic function $f : \mathcal{C} \to \mathbb{R}$ defined by $f(x) = \langle Ax, x\rangle$, we will simply use the notation f for the function $\tilde{f} : \mathcal{D} \to \mathbb{R}$ defined by $\tilde{f}(x) = \langle Ax, x\rangle$.

3 Quadratic functions on spherical convex sets

In this section we present some general properties satisfied by quadratic functions which are spherically convex.

Proposition 3. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a proper cone, $\mathcal{C} = \mathrm{int}(\mathcal{K}) \cap \mathbb{S}$ and let $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A \in \mathbb{R}^{n\times n}$. Then, the following statements are equivalent:*

(i) The function f is spherically convex;

(ii) $\langle Ax, x\rangle - \langle Ay, y\rangle \geq 0$, for all $x \in \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ with $x \perp y$.

Proof. To prove the equivalence of items (i) and (ii), note that $\mathcal{C} = \mathrm{int}(\mathcal{K}) \cap \mathbb{S}$ is an open spherically convex set, $Df(x) = 2Ax$ and $D^2 f(x) = 2A$, for all $x \in \mathcal{C}$. Then, from item (iii) of Proposition 2 we conclude that $\langle Ax, x\rangle \geq \langle Ay, y\rangle$, for all $x \in \mathbb{S}$ and $y \in \mathcal{C}$ with $x \perp y$. Hence, by continuity this inequality extends for all $y \in \mathcal{K} \cap \mathbb{S}$ with $x \perp y$. □

Remark 2. *Let $x, y \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Then, it is easy to see that $\langle Ax, x\rangle \geq \langle Ay, y\rangle$ if and only if $\langle (A + A^T)x, x\rangle \geq \langle (A + A^T)y, y\rangle$. Thus, from Proposition 3 we conclude that $f(x) = \langle Ax, x\rangle$ is spherically convex if and only if $h(x) = \langle (A + A^T)x, x\rangle$ is spherically convex. Therefore without loss of generality, for the purpose of studying the convexity of the function $f(x) = \langle Ax, x\rangle$, we can assume that A is a symmetric matrix.*

Proposition 4. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a proper cone, $\mathcal{C} = \mathrm{int}(\mathcal{K}) \cap \mathbb{S}$ and let $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. The following statements are equivalent:*

(i) The function f is spherically convex;

(ii) $2\langle Ax, y\rangle \leq (\langle Ax, x\rangle + \langle Ay, y\rangle)\langle x, y\rangle$, for all $x, y \in \mathcal{K} \cap \mathbb{S}$.

As a consequence, if $\mathcal{K}$ is superdual and f is spherically convex, then A has the $\mathcal{K}$-Z-property.

Proof. The equivalence of items (i) and (ii) follows from item (ii) of Proposition 2 after some algebraic manipulations, by using arguments similar to the ones used in the proof of the equivalence of items (i) and (ii) in Proposition 3. For the second part, let $x \in \mathcal{K} \cap \mathbb{S}$ and $y \in \mathcal{K}^* \cap \mathbb{S} \subset \mathcal{K} \cap \mathbb{S}$ with $x \perp y$. Since f is spherically convex and $x \perp y$, the inequality in item (ii) implies $\langle Ax, y\rangle \leq 0$. Therefore, the result follows from the definition of $\mathcal{K}$-Z-property. □

Proposition 5. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a superdual proper cone, $\mathcal{C} = \mathrm{int}(\mathcal{K}) \cap \mathbb{S}$ and $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. If f is spherically convex, then the following statements hold:*

(i) If $x, y \in (\mathcal{K} \cup -\mathcal{K}) \cap \mathbb{S}$ such that $x \perp y$, then $\langle Ax, x\rangle = \langle Ay, y\rangle$;

(ii) If $x \in \mathrm{int}(\mathcal{K}) \cap \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ such that $x \perp y$, then $Ax \perp y$;

(iii) If $x \in -\mathrm{int}(\mathcal{K}) \cap \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ such that $x \perp y$, then $Ax \perp y$.

Proof. For proving item (i), we use the equivalence of items (i) and (ii) of Proposition 3 to obtain that $\langle Ax, x\rangle \geq \langle Ay, y\rangle$ and $\langle Ay, y\rangle \geq \langle Ax, x\rangle$, for all $x, y \in (\mathcal{K} \cup -\mathcal{K}) \cap \mathbb{S}$, and the results follows. To prove item (ii), given $x \in \mathrm{int}(\mathcal{K}) \cap \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ such that $x \perp y$, define $u = (1/(m^2+1))(mx - y)$ and $v = (1/(m^2 + 1))(x + my)$, where m is a positive integer. Since $x \in \mathrm{int}(\mathcal{K}) \cap \mathbb{S}$, if m is large enough, then $(1/m)u \in \mathcal{K}$ and therefore $u \in \mathcal{K}$ too. It is easy to check that $u, v \in \mathcal{K} \cap \mathbb{S}$ such that $u \perp v$. By using item (i) twice, we conclude $\langle mAx - Ay, mx - y\rangle = \langle Ax + mAy, x + my\rangle$, which after some algebraic transformations, bearing in mind that $A = A^T$, implies $Ax \perp y$. We can prove item (iii) in a similar fashion. □

Corollary 1. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a strongly superdual proper cone, $\mathcal{C} = \mathrm{int}(\mathcal{K}) \cap \mathbb{S}$ and let $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. If f is spherically convex, then A is $\mathcal{K}$-Lyapunov-like.*

Proof. Let $x \in \mathcal{K} \cap \mathbb{S}$ and $y \in \mathcal{K}^* \cap \mathbb{S} \subset \mathrm{int}(\mathcal{K}) \cap \mathbb{S}$ with $x \perp y$. Then, item (ii) of Proposition 5 implies $Ax \perp y$ and the result follows from the definition of the K-Lyapunov-like property. $\square$

Proposition 6. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a superdual proper cone, $\mathcal{C} = \mathrm{int}(\mathcal{K}) \cap \mathbb{S}$ and $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. If A is $\mathcal{K}$-copositive and f is spherically convex, then A is positive semidefinite.*

Proof. Since A is $\mathcal{K}$-copositive we have $\langle Ax, x\rangle \geq 0$ for all $x \in (\mathcal{K}^* \cup -\mathcal{K}^*) \cap \mathbb{S} \subset (\mathcal{K} \cup -\mathcal{K}) \cap \mathbb{S}$. Assume that $x \in \mathbb{S} \setminus (\mathcal{K}^* \cup -\mathcal{K}^*)$. Then, there exists an $y \in \mathcal{K} \cap \mathbb{S}$ such that $y \perp x$. Suppose that there is no such y. Then, we must have that either $\langle u, x\rangle < 0$ for all $u \in \mathcal{K} \setminus \{0\}$, or $\langle u, x\rangle > 0$ for all $u \in \mathcal{K} \setminus \{0\}$. Otherwise, if there is an $u \in \mathcal{K} \setminus \{0\}$ with $\langle u, x\rangle < 0$ and a $v \in \mathcal{K} \setminus \{0\}$ with $\langle v, x\rangle \geq 0$, then $\psi(0) < 0$ and $\psi(1) \geq 0$, where the continuous function $\psi : \mathbb{R} \to \mathbb{R}$ is defined by $\psi(t) = \langle (1-t)u + tv, x\rangle$. Hence, there is an $s \in [0, 1]$ such that $\psi(s) = 0$. By the convexity of $\mathcal{K} \setminus \{0\}$ ($\mathcal{K} \setminus \{0\}$ is spherically convex because $\mathcal{K}$ is pointed), we conclude that $(1-s)u + sv \in \mathcal{K} \setminus \{0\}$. Let $w = (1-s)u + sv$ and $y = w/\|w\|$. Clearly, $y \in \mathcal{K} \cap \mathbb{S}$ and $y \perp x$, which contradicts our assumptions. If $\langle u, x\rangle < 0$ for all $u \in \mathcal{K} \setminus \{0\}$, then $x \in -\mathcal{K}^*$, which is a contradiction. If $\langle u, x\rangle > 0$ for all $u \in \mathcal{K} \setminus \{0\}$, then $x \in \mathcal{K}^*$, which is also a contradiction. Thus, there exists an $y \in \mathcal{K} \cap \mathbb{S}$ such that $y \perp x$. Since f is convex, Proposition 3 implies that $\langle Ax, x\rangle \geq \langle Ay, y\rangle$. Since A is K-copositive, we have $\langle Ay, y\rangle \geq 0$ and hence $\langle Ax, x\rangle \geq 0$. Thus, $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{S}$. In conclusion, A is positive semidefinite. $\square$

By using arguments similar to the ones used in the proof of Proposition 6 we can also prove the following result.

Proposition 7. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a subdual proper cone, $\mathcal{C} = \mathrm{int}(\mathcal{K}) \cap \mathbb{S}$ and $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. If A is $\mathcal{K}^*$-copositive and f is spherically convex, then A is positive semidefinite.*

4 Quadratic functions on spherical positive orthant

In this section we present a condition characterizing the spherical convexity of quadratic functions on the spherical convex set associated to the positive orthant cone.

Theorem 1. *Let $\mathcal{C} = \mathbb{S} \cap \mathbb{R}^n_{++}$ and $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. Then, f is spherically convex if and only if there exists $\lambda \in \mathbb{R}$ such that $A = \lambda I_n$. In this case, f is a constant function.*

Proof. Assume that there exists $\lambda \in \mathbb{R}$ such that $A = \lambda I_n$. In this case, $f(x) = \lambda$, for all $x \in \mathcal{C}$. Obviously, any constant function is spherically convex. Conversely, suppose that f is convex. From the equivalence of items (i) and (ii) of Proposition 3 we have

$$\langle Ax, x\rangle \geq \langle Ay, y\rangle, \tag{2}$$

for any $y \in \mathbb{R}^n_+$ and any $x \perp y$ with $x, y \in \mathbb{S}$. First take $x = e^i$ and $y = e^j$. Then, (2) implies that $a_{jj} \geq a_{ii}$. Hence, by swapping i and j, we conclude that $a_{ii} = \lambda$ for any i, where $\lambda \in \mathbb{R}$ is a constant. Next take $y = (1/\sqrt{2})(e^i + e^j)$ and $x = (1/\sqrt{2})(e^i - e^j)$. This leads to $a_{ij} \leq 0$, for any

i, j. Hence, $A = B + \lambda I_n$, where B is a Z-matrix with zero diagonal. It is easy to see that equation (2) is equivalent to

$$\langle Bx, x\rangle \geq \langle By, y\rangle, \tag{3}$$

for any $y \geq 0$ and any $x \perp y$ with $x, y \in \mathbb{S}$. Let i, j be arbitrary but different and k different from both i and j. Let $y = e^k$ and $x = (1/\sqrt{2})(e^i + e^j)$. Then, (3) implies that $a_{ij} = b_{ij} \geq 0$. Together with $a_{ij} \leq 0$ this gives $a_{ij} = b_{ij} = 0$. Hence $A = \lambda I_n$ and therefore $f(x) = \lambda$, for any $x \in \mathcal{C}$, and the proof is concluded. □

5 Quadratic functions on circular spherical convex sets

In section we present a condition characterizing the spherical convexity of quadratic functions on spherical convex sets associated to the circular cones. We begin with the following definition: Let $\mathcal{L} \subset \mathbb{R}^n$ be the *Lorentz cone* defined by

$$\mathcal{L} := \left\{ x \in \mathbb{R}^n \ : \ x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2} \right\}. \tag{4}$$

Lemma 2. *Let $\mathcal{L}$ be the Lorentz cone, $x := (x_1, \tilde{x})$ and $y := (y_1, \tilde{y})$ in $\mathbb{S}$. Then the following statements hold:*

(i) $y \in -\mathcal{L} \cup \mathcal{L}$ if and only if $y_1^2 \geq 1/2$. Moreover, $y_1^2 \geq 1/2$ if and only if $\|\tilde{y}\|^2 \leq 1/2$;

(ii) $y \in -\operatorname{int}(\mathcal{L}) \cup \operatorname{int}(\mathcal{L})$ if and only if $y_1^2 > 1/2$. Moreover, $y_1^2 > 1/2$ if and only if $\|\tilde{y}\|^2 < 1/2$;

(iii) $x \notin -\operatorname{int}(\mathcal{L}) \cup \operatorname{int}(\mathcal{L})$ if and only if $x_1^2 \leq 1/2$. Moreover, $x_1^2 \leq 1/2$ if, and only if, $\|\tilde{x}\|^2 \geq 1/2$;

(iv) If $y \in -\mathcal{L} \cup \mathcal{L}$ and $x \perp y$ then $x \notin -\operatorname{int}(\mathcal{L}) \cap \operatorname{int}(\mathcal{L})$. Moreover, $x \notin -\operatorname{int}(\mathcal{L}) \cap \operatorname{int}(\mathcal{L})$ if, and only if $x_1^2 \leq 1/2$. Furthermore, $x_1^2 \leq 1/2$ if and only if $\|\tilde{x}\|^2 \geq 1/2$.

Proof. Items (i)-(iii) follow easily from the definitions of $\mathbb{S}$ and $\mathcal{L}$. Item (iv) follows from Lemma 1 and item (iii). □

Remark 3. *Let $\tilde{Q} \in \mathbb{R}^{(n-1)\times(n-1)}$ be an orthogonal matrix. Then, $Q = \operatorname{diag}(1, \tilde{Q})$ is also ortogonal and $Q\mathcal{L} = \mathcal{L}$. Hence, from Remark 1 we conclude that $f : \mathcal{L} \cap \mathbb{S} \to \mathbb{R}$ is spherically convex if, and only if, $g := f \circ Q = \mathcal{L} \cap \mathbb{S} \to \mathbb{R}$ is spherically convex.*

Theorem 2. *Let $\mathcal{C} = \operatorname{int}(\mathcal{L}) \cap \mathbb{S}$ and $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. Then f is spherically convex if and only if there exist $a, \lambda \in \mathbb{R}$ with $\lambda \geq a$ such that $A = \operatorname{diag}(a, \lambda I_{n-1})$.*

Proof. Assume that f is spherically convex. Let $x, y \in \mathcal{L} \cap \mathbb{S}$ with $x \perp y$ be defined by

$$x = \frac{1}{\sqrt{2}}e^1 + \frac{1}{\sqrt{2}}e^i, \qquad y = \frac{1}{\sqrt{2}}e^1 - \frac{1}{\sqrt{2}}e^i, \qquad i \in \{2, \dots, n\}.$$

Hence the item (i) of Proposition 5 implies that $\langle Ax, x\rangle = \langle Ay, y\rangle$. Hence, after carrying out these inner products, we obtain

$$\frac{1}{2}(a_{11} + a_{1i}) + \frac{1}{2}(a_{i1} + a_{ii}) = \frac{1}{2}(a_{11} - a_{1i}) - \frac{1}{2}(a_{i1} - a_{ii}), \qquad i \in \{2, \dots, n\}.$$

Since A is a symmetric matrix, the last equality implies that $a_{1i} = 0$, for all $i \in \{2, \dots, n\}$. Thus, by letting $a = a_{11}$, we have $A = \operatorname{diag}(a, \tilde{A})$ with $\tilde{A} \in \mathbb{R}^{(n-1)\times(n-1)}$ a symmetric matrix. Let

$\tilde{Q} \in \mathbb{R}^{(n-1)\times(n-1)}$ be an orthogonal matrix such that $\tilde{Q}^T \tilde{A} \tilde{Q} = \Lambda$, where $\Lambda = \mathrm{diag}(\lambda_2, \ldots, \lambda_n)$ and λ_i is an eigenvalue of $\tilde{A}$, for all $i \in \{2, \ldots, n\}$. Thus, Remark 3 implies that $f : \mathcal{L} \cap \mathbb{S} \to \mathbb{R}$ is spherically convex if, and only if, $g(x) = \langle \mathrm{diag}(a_{11}, \Lambda)x, x\rangle$ is spherically convex. On the other hand, using Proposition 3 we conclude that $g(x) = \langle \mathrm{diag}(a_{11}, \Lambda)x, x\rangle$ is spherically convex if and only if

$$h(x) = \langle [\mathrm{diag}(a_{11}, \Lambda) - a_{11}I_n]x, x\rangle = \langle [\Lambda - a_{11}I_{n-1}]\tilde{x}, \tilde{x}\rangle, \ x := (x_1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{n-1},$$

is spherically convex. We are going to prove that $a_{11} \leq \lambda_2 = \ldots = \lambda_n$. Since h is spherically convex, from Proposition 3 we have

$$h(x) - h(y) = \langle [\Lambda - a_{11}I_{n-1}]\tilde{x}, \tilde{x}\rangle - \langle [\Lambda - a_{11}I_{n-1}]\tilde{y}, \tilde{y}\rangle \geq 0, \tag{5}$$

for all $x = (x_1, \tilde{x}) \in \mathbb{S}$, $y = (y_1, \tilde{y}) \in \mathcal{L} \cap \mathbb{S}$ with $x \perp y$. Let $x \in \mathbb{S}$ and $y \in \mathcal{L} \cap \mathbb{S}$ with $x \perp y$ be defined by

$$x = -\left(\frac{1}{\sqrt{2}}\cos\theta\right)e^1 + \left(\frac{1}{2}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)e^i + \left(\frac{1}{2}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)e^j, \tag{6}$$

$$y = \frac{1}{\sqrt{2}}e^1 + \frac{1}{2}e^i + \frac{1}{2}e^j, \tag{7}$$

where $\theta \in (0, \pi)$. From (6) and (7), it is straightforward to check that $x \in \mathbb{S}$, $y \in \mathcal{L} \cap \mathbb{S}$ and $x \perp y$. Hence, (5) becomes

$$\left(\frac{1}{4}\sin^2\theta - \frac{1}{\sqrt{2}}\cos\theta\sin\theta\right)\lambda_i + \left(\frac{1}{4}\sin^2\theta + \frac{1}{\sqrt{2}}\cos\theta\sin\theta\right)\lambda_j \geq 0,$$

or, after simplifying by $\sin\theta \neq 0$, that

$$\left(\frac{1}{4}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta\right)\lambda_i + \left(\frac{1}{4}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right)\lambda_j \geq 0.$$

By tending with θ to zero in this latter inequality, we obtain $\lambda_j \geq \lambda_i$. Hence, by swapping i and j in (6) and (7) we can also prove that $\lambda_i \geq \lambda_j$, and then $\lambda_i = \lambda_j$, for all $i, j \neq 1$. Therefore, $\lambda_2 = \ldots = \lambda_n$ and $B = \lambda I_{n-1}$ and $A = \mathrm{diag}(a, \lambda I_{n-1})$, where $a := a_{11}$ and $\lambda := \lambda_2 = \cdots = \lambda_n$. Hence, $\Lambda = \lambda I_{n-1}$ and then (5) becomes $[\lambda - a_{11}][\|\tilde{x}\|^2 - \|\tilde{y}\|^2] \geq 0$. Bearing in mind that $\mathcal{L} = \mathcal{L}^*$, Lemma 2 implies $\|\tilde{x}\|^2 - \|\tilde{y}\|^2 \geq 0$, and then we have from the previous inequality that $a = a_{11} \leq \lambda$. Conversely, assume that $A = \mathrm{diag}(a, \lambda I_{n-1})$ and $\lambda \geq a$. Then $f(x) = \langle [\mathrm{diag}(a, \lambda I_{n-1}]x, x\rangle$ and Proposition 3 implies that f is spherically convex if, and only if,

$$h(x) = \langle [\mathrm{diag}(a, \lambda I_{n-1}) - aI_n]x, x\rangle = \langle [\lambda - a]I_{n-1}\tilde{x}, \tilde{x}\rangle, \qquad x := (x_1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{n-1},$$

is spherically convex. Take $x = (x_1, \tilde{x}) \in \mathbb{S}$ and $y = (y_1, \tilde{y}) \in \mathcal{L} \cap \mathbb{S}$ with $x \perp y$. Thus, from Lemma 1 and (4) we have $\|\tilde{x}\|^2 \geq \|\tilde{y}\|^2$. Hence considering that $a \leq \lambda$ we conclude that

$$\langle [\lambda - a]I_{n-1}\tilde{x}, \tilde{x}\rangle - \langle [\lambda - a]I_{n-1}\tilde{y}, \tilde{y}\rangle = [\lambda - a][\|\tilde{x}\|^2 - \|\tilde{y}\|^2] \geq 0.$$

Therefore, Proposition 3 implies that h is spherically convex and then f is also spherically convex. □

Remark 4. *Assume that f in Theorem 2 is spherically convex in $\mathcal{L}\cap\mathbb{S}$. Hence there exist $a,\lambda\in\mathbb{R}$ with $\lambda\geq a$ such that $A=\operatorname{diag}(a,\lambda I_{n-1})$ and then $f(x)=ax_1^2+\lambda\|\tilde{x}\|^2=\lambda-(\lambda-a)x_1^2$, where $x:=(x_1,\tilde{x})\in\mathcal{L}\cap\mathbb{S}$. Hence, it is clear that the minimum of f on $\mathcal{L}\cap\mathbb{S}$ is obtained when x_1 is maximal, that is, when $x_1=1$, which happens exactly when $x=e^1$. Similarly, the maximum of f on $\mathcal{L}\cap\mathbb{S}$ is obtained when x_1 is minimal, that is, when $x_1=1/\sqrt{2}$ (see item (i) of Lemma 2), which happens exactly when $\|\tilde{x}\|=x_1=1/\sqrt{2}$. Hence,*

$$\operatorname{argmin}\{f(x):x\in\mathcal{L}\cap\mathbb{S}\}=e^1,\quad \min\{f(x):x\in\mathcal{L}\cap\mathbb{S}\}=a,$$

$$\operatorname{argmax}\{f(x):x\in\mathcal{L}\cap\mathbb{S}\}=\left\{\frac{1}{\sqrt{2}}(1,\hat{x})\in\mathbb{R}\times\mathbb{R}^{n-1}:\|\hat{x}\|=1\right\},\quad \max\{f(x):x\in\mathcal{L}\cap\mathbb{S}\}=\frac{a+\lambda}{2}.$$

Remark 5. *If $\lambda>a$ then Theorem 2 implies that $f(x)=\langle\operatorname{diag}(a,\lambda I_{n-1})x,x\rangle$ is spherically convex. However, in this case $\operatorname{diag}(a,\lambda,\ldots,\lambda)$ does not have the $\mathcal{L}$-Lyapunov-like property. Hence, Corollary 1 is not true if we only require that the cone is superdual proper. Indeed, the Lorentz cone $\mathcal{L}$ is self-dual proper, i.e., $\mathcal{L}^*=\mathcal{L}$ and consequently is superdual proper. Moreover, letting $x,y\in\mathcal{L}\cap\mathbb{S}$ with $x\perp y$ be defined by*

$$x=\frac{1}{\sqrt{2}}e^1+\frac{1}{\sqrt{2}}e^i,\qquad y=\frac{1}{\sqrt{2}}e^1-\frac{1}{\sqrt{2}}e^i,\qquad i\in\{2,\ldots,n\},$$

we have $\langle\operatorname{diag}(a,\lambda I_{n-1})x,y\rangle=(a-\lambda)/2<0$. Therefore, $\operatorname{diag}(a,\lambda I_{n-1})$ does not have the $\mathcal{L}$-Lyapunov-like property, and the strong superduality of the cone is necessary in Corollary 1.

Let $\alpha\in(0,\ \pi/4)$ and $\mathcal{L}_\alpha\subset\mathbb{R}^n$ be the *circular cone* defined by

$$\mathcal{L}_\alpha:=\left\{x\in\mathbb{R}^n\ :\ x_1\geq\tan\alpha\sqrt{x_2^2+\cdots+x_n^2}\right\}. \tag{8}$$

Theorem 3. *Let $\alpha\in(0,\ \pi/4)$, $\mathcal{C}_\alpha=\operatorname{int}(\mathcal{L}_\alpha)\cap\mathbb{S}$ and $f:\mathcal{C}_\alpha\to\mathbb{R}$ be defined by $f(x)=\langle Ax,x\rangle$, where $A=A^T\in\mathbb{R}^{n\times n}$. Then, f is spherically convex if and only if there exist $a,\lambda\in\mathbb{R}$ with $\lambda\geq a$ such that $A=\operatorname{diag}(a,\lambda I_{n-1})$.*

Proof. Suppose that there exist $a,\lambda\in\mathbb{R}$ with $\lambda\geq a$ such that $A=\operatorname{diag}(a,\lambda I_{n-1})$. Since $\mathcal{L}_\alpha\subset\mathcal{L}$, where $\mathcal{L}$ is the Lorentz cone, we have $\mathcal{C}_\alpha\subset\mathcal{C}$, where $=\operatorname{int}(\mathcal{L})\cap S$. Hence, by using Theorem 2, we conclude that f is spherically convex. Conversely, suppose that f is spherically convex. Define

$$A:=\begin{bmatrix}a_{11} & b^T\\ b & B\end{bmatrix},\qquad u:=\begin{bmatrix}\sin\alpha\\ \cos\alpha\ \hat{x}\end{bmatrix},\qquad v:=\begin{bmatrix}\sin\alpha\\ -\cos\alpha\ \hat{x}\end{bmatrix},\qquad \|\hat{x}\|=1,\quad \hat{x}\in\mathbb{R}^{n-1}. \tag{9}$$

By using the definition of $\mathcal{L}_\alpha$ we can prove that $u,v\in\mathcal{L}_\alpha\cap\mathbb{S}$. Moreover, the following equalities hold

$$\begin{aligned}\langle Au,u\rangle+\langle Av,v\rangle&=2[a_{11}\sin^2\alpha+\cos^2\alpha\langle B\hat{x},\hat{x}\rangle],\\ \langle Au,v\rangle&=a_{11}\sin^2\alpha-\cos^2\alpha\langle B\hat{x},\hat{x}\rangle,\\ \langle u,v\rangle&=-\cos(2\alpha).\end{aligned}$$

Hence, by using the inequality in the item (iii) of Proposition 4, we conclude that

$$2\left[a_{11}\sin^2\alpha-\cos^2\alpha\langle B\hat{x},\hat{x}\rangle\right]\leq 2\left[a_{11}\sin^2\alpha+\cos^2\alpha\langle B\hat{x},\hat{x}\rangle\right][-\cos(2\alpha)]$$

After some algebra, the last inequality becomes $2\cos^2\alpha\sin^2\alpha\langle B\hat{x},\hat{x}\rangle \geq 2a_{11}\cos^2\alpha\sin^2\alpha$. Thus, by using $\alpha \in (0,\ \pi/4)$, we conclude that $\langle B\hat{x},\hat{x}\rangle \geq a_{11}$, for any $\hat{x} \in \mathbb{R}^{n-1}$ with $\|\hat{x}\| = 1$. Therefore, $B - a_{11}I_{n-1}$ is a positive semidefinite matrix. Let

$$x := \begin{bmatrix} \cos\alpha \\ \sin\alpha\hat{x} \end{bmatrix}, \qquad \|\hat{x}\| = 1, \quad \hat{x} \in \mathbb{R}^{n-1}. \tag{10}$$

Thus, by using the definitions of A and v in (9), and x in (10), we conclude that

$$\langle Ax, x\rangle = a_{11}\cos^2\alpha - 2\sin\alpha\cos\alpha\langle b,\hat{x}\rangle + \sin^2\alpha\langle B\hat{x},\hat{x}\rangle,$$
$$\langle Av, v\rangle = a_{11}\sin^2\alpha + 2\sin\alpha\cos\alpha\langle b,\hat{x}\rangle + \cos^2\alpha\langle B\hat{x},\hat{x}\rangle,$$

besides $x \in \mathbb{S}$, $v \in \mathcal{L}_\alpha \cap \mathbb{S}$ and $x \perp v$. Hence, by using item (ii) of Proposition 3, after some algebraic manipulations and taking into account that $B - a_{11}I_{n-1}$ is a positive semidefinite matrix, we have

$$-2\sin(2\alpha)\langle b,\hat{x}\rangle \geq \cos(2\alpha)(\langle B\hat{x},\hat{x}\rangle - a_{11}) \geq 0.$$

Since $\alpha \in (0,\ \pi/4)$, from the last inequality we have $\langle b,\hat{x}\rangle \leq 0$ for any $\hat{x} \in \mathbb{R}^{n-1}$ with $\|\hat{x}\| = 1$. Therefore, $b = 0$ and then

$$A = \operatorname{diag}(a_{11}, B), \qquad B = B^T.$$

Let $\tilde{Q} \in \mathbb{R}^{(n-1)\times(n-1)}$ be an orthogonal matrix such that $\tilde{Q}^T B\tilde{Q} = \Lambda$, where $\Lambda = \operatorname{diag}(\lambda_2,\ldots,\lambda_n)$ and λ_i is an eigenvalue of B, for all $i \in \{2,\ldots,n\}$. Thus, Remark 3 implies that $f : L \to \mathbb{R}$ is spherically convex if, and only if, $g(x) = \langle\operatorname{diag}(a_{11},\Lambda)x, x\rangle$ is spherically convex. On the other hand, by using Proposition 3, we conclude that $g(x) = \langle\operatorname{diag}(a_{11},\Lambda)x, x\rangle$ is spherically convex if and only if

$$h(x) = \langle[\operatorname{diag}(a_{11},\Lambda) - a_{11}I_n]x, x\rangle = \langle[\Lambda - a_{11}I_{n-1}]\tilde{x},\tilde{x}\rangle,\ x := (x_1,\tilde{x}) \in \mathbb{R}\times\mathbb{R}^{n-1},$$

is spherically convex. We are going to prove that $a_{11} \leq \lambda_2 = \ldots = \lambda_n$. Since h is spherically convex, from Proposition 3, we have

$$h(x) - h(y) = \langle[\Lambda - a_{11}I_{n-1}]\tilde{x},\tilde{x}\rangle - \langle[\Lambda - a_{11}I_{n-1}]\tilde{y},\tilde{y}\rangle \geq 0, \tag{11}$$

for all $x = (x_1,\tilde{x}) \in \mathbb{S}$, $y = (y_1,\tilde{y}) \in \mathcal{L}\cap\mathbb{S}$, with $x \perp y$. Let $x \in \mathbb{S}$ and $y \in \mathcal{L}_\alpha \cap \mathbb{S}$ with $x \perp y$ be defined by

$$x = -(\cos\alpha\cos\varphi)e^1 + (\sin\alpha\cos\theta\cos\varphi - \sin\theta\sin\varphi)e^i + (\sin\alpha\sin\theta\cos\varphi + \cos\theta\sin\varphi)e^j, \tag{12}$$
$$y = (\sin\alpha)e^1 + (\cos\alpha\cos\theta)e^i + (\cos\alpha\sin\theta)e^j, \tag{13}$$

where $i \neq j$, $\theta = \pi/4$ and $\varphi \in (0,\pi)$ with $\cos\varphi = \tan\alpha$. It is straightforward to check that $x \in \mathbb{S}$ and $y \in \mathcal{L}\cap\mathbb{S}$ with $x \perp y$. Hence (11) becomes

$$\left[(\sin\alpha\cos\theta\cos\varphi - \sin\theta\sin\varphi)^2 - (\cos\alpha\cos\theta)^2\right](\lambda_i - a_{11}) + \left[(\sin\alpha\sin\theta\cos\varphi + \cos\theta\sin\varphi)^2 - (\cos\alpha\sin\theta)^2\right](\lambda_j - a_{11}) \geq 0. \tag{14}$$

Since $\theta = \pi/4$ and $\varphi \in (0,\pi)$ with $\cos\varphi = \tan\alpha$, after some calculations, we conclude that

$$(\sin\alpha\cos\theta\cos\varphi - \sin\theta\sin\varphi)^2 - (\cos\alpha\cos\theta)^2 = -2\sin\alpha\sin\theta\cos\theta\sin\varphi\cos\varphi < 0,$$
$$(\sin\alpha\sin\theta\cos\varphi + \cos\theta\sin\varphi)^2 - (\cos\alpha\sin\theta)^2 = 2\sin\alpha\sin\theta\cos\theta\sin\varphi\cos\varphi > 0.$$

Then, (14) reduces to $-(\lambda_i - a_{11}) + (\lambda_j - a_{11}) \geq 0$, which implies that $\lambda_j \geq \lambda_i$. By swapping i and j in (12) and (13) we can also prove that $\lambda_i \geq \lambda_j$, and then $\lambda_i = \lambda_j$, for all $i,j \neq 1$. In conclusion $\lambda_2 = \cdots = \lambda_n$ and therefore $B = \lambda I_{n-1}$ and $A = \operatorname{diag}(a, \lambda I_{n-1})$, where $a := a_{11}$ and $\lambda := \lambda_2 = \cdots = \lambda_n$. By using that $B - a_{11}I_{n-1} = (\lambda - a)I_{n-1}$ is positive semidefinite, we obtain that $\lambda \geq a$ and the proof is concluded. □

Corollary 2. *Let $\alpha \in (0, \ \pi/4]$, $\mathcal{C}_\alpha = \mathrm{int}(\mathcal{L}_\alpha) \cap \mathbb{S}$ and $f : \mathcal{C}_\alpha \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. Then, f is convex if and only if there exist $\lambda, b \in \mathbb{R}$ with $b \geq 0$ such that $f(x) = \lambda - bx_1^2$, where $x := (x_1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$.*

Proof. First note that $\mathcal{L}_{\pi/4} = \mathcal{L}$. By Theorems 2 and 3 f is spherically convex if and only if there exist $a, \lambda \in \mathbb{R}$ with $\lambda \geq a$ such that $A = \mathrm{diag}(a, \lambda I_{n-1})$. Hence, f is convex if and only if there exist $a, \lambda \in \mathbb{R}$ with $\lambda \geq a$ such that

$$f(x) = ax_1^2 + \lambda(x_2^2 + \cdots + x_n^2) = ax_1^2 + \lambda(1 - x_1^2) = \lambda - (\lambda - a)x_1^2, \qquad x := (x_1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

Therefore, by letting $b = \lambda - a$, the result follows from the last equivalence. □

Corollary 3. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a proper cone, $\mathcal{C} = \mathrm{int}(\mathcal{K}) \cap \mathbb{S}$ and $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. If f is spherically convex then A has two eigenvalues $a, \lambda \in \mathbb{R}$ with $\lambda \geq a$. Moreover, a has multiplicity 1 and λ has multiplicity $n-1$.*

Proof. Since $\mathcal{K} \subset \mathbb{R}^n$ is a proper cone, there exist $\alpha \in (0, \ \pi/4)$ and $Q \in \mathbb{R}^{n\times n}$ an orthogonal matrix such that $\tilde{\mathcal{C}}_\alpha := Q\mathcal{C}_\alpha \subset \mathcal{C}$, where $\mathcal{C}_\alpha = \mathrm{int}(\mathcal{L}_\alpha) \cap \mathbb{S}$. Hence, if f is spherically convex in $\mathcal{C}$, then from Remark 1 we conclude that $h(x) = f(Qx) = \langle Q^TAQx, x\rangle$ is spherically convex in $\mathcal{C}_\alpha$. Therefore, from Theorem 3 we have that there exist $a, \lambda \in \mathbb{R}$ with $\lambda \geq a$ such that $Q^TAQ = \mathrm{diag}(a, \lambda I_{n-1})$ and the result follows. □

Remark 6. *Theorem 1 shows that the converse of Corollary 3 does not hold.*

6 Final remarks

This paper is a continuation of [5,6], where we studied some basic intrinsic properties of spherically convex functions on spherically convex sets of the sphere. We expect that the results of this paper can aid in the understanding of the behaviour of spherically convex functions on spherically convex sets of the sphere. In the future we will also study the problem of determining the spherically quasiconvex quadratic and more general spherically quasiconvex functions [21] (see also [15] for the definition of quasiconvex functions) on spherically convex sets of the sphere. As far as we know this would be the first study of spherically quasiconvex functions. As an even more challenging problem, we will work towards developing efficient algorithms for minimizing quadratic and more general functions on spherically convex sets of the sphere. We foresee further progress in these topics in the nearby future.

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