Projection onto simplicial cones by Picard's method *

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Abstract

By using Moreau's decomposition theorem for projecting onto cones, the problem of projecting onto a simplicial cone is reduced to finding the unique solution of a nonsmooth system of equations. It is shown that Picard's method applied to the system of equations associated to the problem of projecting onto a simplicial cone generates a sequence that converges linearly to the solution of the system. Numerical experiments are presented making the comparison between Picard's and semi-smooth Newton's methods to solve the nonsmooth system associated with the problem of projecting a point onto a simplicial cone.

1 Introduction

The interest in the subject of projection arises in several situations, having a wide range of applications in pure and applied mathematics such as Convex Analysis (see e.g. [23]), Optimization (see e.g. [4, 9, 10, 20, 41, 44]), Numerical Linear Algebra (see e.g. [42]), Statistics (see e.g. [6, 15, 24]), Computer Graphics (see e.g. [19]) and Ordered Vector Spaces (see e.g. [1, 26, 27, 35, 37, 38]). More specifically, the projection onto a polyhedral cone, which has as a special case the projection onto a simplicial one, is a problem of high impact on scientific community[1]. The geometric nature of this problem makes it particularly interesting and important in many areas of science and technology such as Statistics (see e.g. [24]), Computation (see e.g. [25]), Optimization (see e.g. [32, 44]) and Ordered Vector Spaces (see e.g. [35]).

The projection onto a general simplicial cone is difficult and computationally expensive, this problem has been studied e.g. in [2, 16, 20, 34, 35, 44]. It is a special convex quadratic program and

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[1] see the popularity of the Wikimization page Projection on Polyhedral Cone at http://www.convexoptimization.com/wikimization/index.php/Special:Popularpages

its KKT optimality conditions form the linear complementarity problem (LCP) associated with it, see e.g [33,34,44]. Therefore, the problem of projecting onto simplicial cones can be solved by active set methods [5, 29, 30, 33] or any algorithms for solving LCPs, see e.g [5, 33] and special methods based on its geometry, see e.g [33, 34]. Other fashionable ways to solve this problem are based on the classical von Neumann algorithm (see e.g. the Dykstra algorithm [13, 15, 48]). Nevertheless, these methods are also quite expensive (see the numerical results in [32] and the remark preceding section 6.3 in [43]).

In this paper we particularize the Moreau's decomposition theorem for simplicial cones. This leads to an equivalence between the problem of projecting a point onto a simplicial cone and one of finding the unique solution of a nonsmooth system of equations. We apply Picard's method to find a unique solution of the obtained associated system. Under a mild assumption on the simplicial cone we show that the method generate a sequence that converges linearly to the solution of the associated system of equations. Numerical experiments are presented making the comparison between Picard's and semi-smooth Newton's methods for solving the nonsmooth system associated with the problem of projecting a point onto a simplicial cone.

The organization of the paper is as follows. In Section 2, some notations, basic results used in the paper and the statement of the problems that we are interested are presented, in particular, the problem of projecting onto simplicial cone. In Section 3 we present some results about projection onto simplicial cones. In Section 4 we present two different Picard's iterations for solving the problem of projecting onto simplicial cone. In Section 5 theoretical and numerical comparisons between Picard's methods and semi-smooth Newton's method for solving the problem of projecting onto simplicial cone [17] are provided. Some final remarks are made in Section 6.

2 Preliminaries

Consider $\mathbb{R}^m$ endowed with an orthogonal coordinate system and let $\langle \cdot, \cdot \rangle$ be the canonical scalar product defined by it. Denote by $\|\cdot\|$ be the norm generated by $\langle \cdot, \cdot \rangle$. If $a \in \mathbb{R}$ and $x = (x^1, \ldots, x^m) \in \mathbb{R}^m$, then denote $a^+ := \max\{a, 0\}$, $a^- := \max\{-a, 0\}$ and

$$x^+ := \left((x^1)^+, \ldots, (x^m)^+\right), \qquad x^- := \left((x^1)^-, \ldots, (x^m)^-\right), \qquad |x| = \left(|x^1|, \ldots, |x^m|\right).$$

For $x \in \mathbb{R}^m$, the vector $\operatorname{sgn}(x)$ will denote a vector with components equal to 1, 0 or -1 depending on whether the corresponding component of the vector x is positive, zero or negative. We will call a closed set $K \subset \mathbb{R}^m$ a *cone* if the following conditions hold:

(i) $\lambda x + \mu y \in K$ for any $\lambda, \mu \geq 0$ and $x, y \in K$,

(ii) $x, -x \in K$ implies $x = 0$.

Let $K \subset \mathbb{R}^n$ be a closed convex cone. The *polar cone* and the *dual cone* of K are, respectively, the sets

$$K^{\perp} := \{x \in \mathbb{R}^n, \mid \langle x, y \rangle \leq 0, \forall\, y \in K\}, \qquad K^* := \{x \in \mathbb{R}^n \mid \langle x, y \rangle \geq 0, \forall\, y \in K\}. \tag{1}$$

It is easy to see that $K^{\perp} = -K^*$. The set of all $m \times m$ real matrices is denoted by $\mathbb{R}^{m \times m}$, I denotes the $m \times m$ identity matrix and $\operatorname{diag}(x)$ will denote a diagonal matrix corresponding to elements of x.

For an $M \in \mathbb{R}^{m\times m}$ consider the norm defined by $\|M\| := \max_{x\neq 0}\{\|Mx\| \ : \ x \in \mathbb{R}^n, \ \|x\| = 1\}$, this definition implies

$$\|Mx\| \leq \|M\|\|x\|, \qquad \|LM\| \leq \|L\|\|M\|, \tag{2}$$

for any $m \times m$ matrices L and M.

Denote $\mathbb{R}^m_+ = \{x = (x^1, \ldots, x^m) \in \mathbb{R}^m : x_1 \geq 0, \ldots, x^m \geq 0\}$ the nonnegative orthant. Let $A \in \mathbb{R}^{m\times m}$ be a nonsingular matrix. Then, the cone

$$K := A\mathbb{R}^m_+ = \{Ax \ : \ x = (x^1, \ldots, x^m) \in \mathbb{R}^m, \ x_1 \geq 0, \ldots, x^m \geq 0\}, \tag{3}$$

is called a *simplicial cone* or *finitely generated cone*. Let $z \in \mathbb{R}^m$, then the *projection* $P_K(z)$ *of the point* z *onto the cone* K is defined by

$$P_K(z) := \text{argmin}\,\{\|z - y\| \ : \ y \in K\}\,.$$

From the definition of simplicial cone associated with the matrix A this definition is equivalent to

$$P_K(z) := \text{argmin}\left\{\frac{1}{2}\|z - Ax\|^2 \ : \ x = (x^1, \ldots, x^m) \in \mathbb{R}^m, \ x_1 \geq 0, \ldots, x^m \geq 0\right\}.$$

Remark 1 *It easy to see that* $P_{\mathbb{R}^m_+}(z) = z^+$. *It is well know that the projection onto a convex set is continuous and nonexpansive, in particular, we have* $\|z^+ - w^+\| \leq \|z - x\|$ *for all* $z, w \in \mathbb{R}^m$,, *see [23].*

The above remark shows that projection onto the nonnegative orthant is an easy problem. On the other hand, the projection onto a general simplicial cone is difficult and computationally expensive, this problem has been studied e.g. in [2, 16, 17, 20, 35, 44]. The statement of the problem that we are interested is:

Problem 1 (projection onto a simplicial cone) *Given* $A \in \mathbb{R}^{m\times m}$ *a nonsingular matrix and* $z \in \mathbb{R}^m$, *find the projection* $P_K(z)$ *of the point* z *onto the simplicial cone* $K = A\mathbb{R}^m_+$.

The problem of projection onto a simplicial cone has many different formulations which allow us develop different techniques for solving them. In the next remark we present some of these formulations.

Remark 2 *Let* $A \in \mathbb{R}^{m\times m}$ *be a nonsingular matrix and* $z \in \mathbb{R}^m$. *From the definition of the the simplicial cone associated with the matrix* A *in* (3), *the problem of projection onto a simplicial cone* $K = A\mathbb{R}^m_+$ *may be stated equivalently as the following quadratic problem*

$$\text{Minimize } \frac{1}{2}\|z - Ax\|^2, \qquad \text{subject to } x \geq 0.$$

Hence, if $v \in R^m$ *is the unique solution of this problem then we have* $P_K(z) = v$. *The above problem is equivalent to the following nonnegative quadratic problem*

$$\text{Minimize } \frac{1}{2}x^\top Qx + x^\top b + c, \qquad \text{subject to } x \geq 0, \tag{4}$$

by taking $Q = A^\top A$, $b = -A^\top z$ and $c = z^\top z/2$. The optimality condition for the problem (4) implies that its solution can be obtained by solving the following linear complementarity problem

$$y = Qx + b, \qquad x \geq 0, \qquad y \geq 0, \qquad x^\top y = 0. \tag{5}$$

where y is a column vector of variables in $\mathbb{R}^m$. It is easy to establish that corresponding to each nonnegative quadratic problems (4) and each linear complementarity problems (5) associated to symmetric positive definite matrixes, there are equivalent problems of projection onto simplicial cones. Therefore, the problem of projecting onto simplicial cones can be solved by active set methods [5, 29, 30, 33] or any algorithms for solving LCPs, see e.g [5, 33] and special methods based on its geometry, see e.g [33, 34]. Other fashionable ways to solve this problem are based on the classical von Neumann algorithm (see e.g. the Dykstra algorithm [13, 15, 48]). Nevertheless, these methods are also quite expensive (see the numerical results in [32] and the remark preceding section 6.3 in [43]).

As we will see in the next section, by using Moreau's decomposition theorem for projecting onto cones, solving Problem 1 is reduced to solving the following problem.

Problem 2 (nonsmooth equation) *Given $A \in \mathbb{R}^{m\times m}$ a nonsingular matrix and $z \in \mathbb{R}^m$, find the unique solution u of the nonsmooth equation*

$$\left(A^\top A - I\right) x^+ + x = A^\top z. \tag{6}$$

In this case, $P_K(z) = Au^+$ where $K = A\mathbb{R}^m_+$.

Since $x^+ = (x + |x|)/2$ the Problem 2 is equivalent to the following problem:

Problem 3 (absolute value equation) *Given $A \in \mathbb{R}^{m\times m}$ a nonsingular matrix and $z \in \mathbb{R}^m$, find the unique solution u of the absolute value equation*

$$\left(A^\top A + I\right) x + \left(A^\top A - I\right) |x| = 2A^\top z. \tag{7}$$

In this case, $P_K(z) = Au^+$ where $K = A\mathbb{R}^m_+$.

We will show in Section 4 that Problem 2 and Problem 3 can be solved by using Picard's method. We end this section with the Banach's fixed point theorem which will be used for proving our main result, its proof can be found in [28] (see Theorem $5.1-2$ pag. 300 and Corollary $5.1-3$ pag. 302).

Theorem 1 (Banach's fixed point theorem) *Let $(\mathbb{X}, d)$ be a non-empty complete metric space, $0 \leq \alpha < 1$ and $T : \mathbb{X} \to \mathbb{X}$ a mapping satisfying $d(T(x), T(y)) \leq \alpha d(x, y)$, for all $x, y \in \mathbb{X}$. Then there exists an unique $x \in \mathbb{X}$ such that $T(x) = x$. Furthermore, x can be found as follows: start with an arbitrary element $x_0 \in \mathbb{X}$ and define a sequence $\{x_n\}$ by $x_{n+1} = F(x_n)$, then $\lim_{n\to+\infty} x_n = x$ and the following inequalities hold:*

$$d(x, x_{n+1}) \leq \frac{\alpha}{1-\alpha} d(x_{n+1}, x_n), \qquad d(x, x_{n+1}) \leq \alpha d(x, x_n), \qquad n = 0, 1, \ldots.$$

3 Moreau's decomposition theorem for simplicial cones

In this section we present some results about projection onto simplicial cones. We recall the following result due to Moreau [31]:

Theorem 2 (Moreau's decomposition theorem) *Let $K, L \subseteq \mathbb{R}^m$ be two mutually polar cones in $\mathbb{R}^m$. Then, the following statements are equivalent:*

(i) $z = x + y,\ x \in K,\ y \in L$ *and* $\langle x, y\rangle = 0$,

(ii) $x = P_K(z)$ *and* $y = P_L(z)$.

Remark 3 *Let K be a cone in $\mathbb{R}^m$. Note that from Moreau's decomposition theorem, definition of the polar cone and the dual cone in* (1) *and the relationship $K^{\perp} = -K^*$ it follows that*

$$P_K(z) = z + P_{K^*}(-z), \qquad \forall\ z \in \mathbb{R}^m.$$

Hence the problem of projecting onto K is equivalent to problem of projecting onto K^*.

The following result follows from the definition of the polar. For a proof see for example [1].

Lemma 1 *Let $A \in \mathbb{R}^{m\times m}$ be a nonsingular matrix. Then,*

$$(A\mathbb{R}^m_+)^{\perp} = -(A^{\top})^{-1}\mathbb{R}^m_+.$$

The following result has been proved in [1] by using Moreau's decomposition theorem and Lemma 1.

Lemma 2 *Let $A \in \mathbb{R}^{m\times m}$ be a nonsingular matrix and $K = A\mathbb{R}^m_+$ the corresponding simplicial cone. Then, for any $z \in \mathbb{R}^m$ there exists a unique $x \in \mathbb{R}^m$ such that the following two equivalent statements hold:*

(i) $z = Ax^+ - (A^{\top})^{-1}x^-,\ x \in \mathbb{R}^m$,

(ii) $Ax^+ = P_K(z)$ *and* $-(A^{\top})^{-1}x^- = P_{K^{\perp}}(z)$.

The following result is a direct consequence of Lemma 2, it shows that solving Problem 1 is reduced to solving Problem 2.

Lemma 3 *Let $A \in \mathbb{R}^{m\times m}$ be a nonsingular matrix, $K = A\mathbb{R}^m_+$ the corresponding simplicial cone and $z \in \mathbb{R}^m$ arbitrary. Then, equations* (6) *and* (7) *have a unique solution u and $P_K(z) = Au^+$, i.e., to solve Problem 1 is equivalent to solving either Problem 2 or Problem 3.*

Proof. Since A is an $m \times m$ nonsingular matrix, multiplying by $A^{\top}$, the equality in item (i) of Lemma 2 is equivalently transformed into

$$A^{\top}Ax^+ - x^- = A^{\top}z.$$

As $-x^- = x - x^+$, the above equality is equivalent to (6). Therefore, equation (6) is equivalent to the equation in item (i) of Lemma 2. Hence, we conclude from Lemma 2 that equation (6) has a unique solution u and $P_K(z) = Au^+$. Since the equations (6) and (7) are equivalent the result follows. □

4 Picard's Method

In this section we will present two different Picard's iterations one on them for solving Problem 2, the other one for solving Problem 3 and as a consequence for solving Problem 1.

4.1 Picard's Method for solving Problem 2

The *Picard's method* for solving Problem 2 is formally defined by

$$x_{k+1} = -\left(A^\top A - I\right) x_k^+ + A^\top z, \qquad k = 0, 1, 2, \dots. \tag{8}$$

The sequence $\{x_k\}$ with starting point $x_0 \in \mathbb{R}^m$, called the *Picard's sequence* for solving Problem 2. The next theorem provides a sufficient condition for the linear convergence of the Picard's iteration (8).

Theorem 3 *Let $A \in \mathbb{R}^{m\times m}$ be a nonsingular matrix, $K = A\mathbb{R}^m_+$ the corresponding simplicial cone and $z \in \mathbb{R}^m$ arbitrary. If*

$$\|A^\top A - I\| < 1, \tag{9}$$

then the Picard's sequence $\{x_k\}$ for solving Problem 2 converges to the unique solution u of equation (6) *from any starting point $x_0 \in \mathbb{R}^m$, $P_K(z) = Au^+$ and the following error bound holds*

$$\|u - x_k\|| \le \frac{\|A^\top A - I\|}{1 - \|A^\top A - I\|}\|x_k - x_{k-1}\|, \qquad \forall\ k = 1, 2 \dots. \tag{10}$$

Moreover, the sequence $\{x_k\}$ converges linearly to u as follows

$$\|u - x_{k+1}\| \le \|A^\top A - I\|\|u - x_k\|, \qquad k = 0, 1, 2, \dots. \tag{11}$$

Proof. Define the function $F : \mathbb{R}^m \to \mathbb{R}^m$ as

$$F(x) = -\left(A^\top A - I\right) x^+ + A^\top z. \tag{12}$$

Since Remark 1 implies $\|x^+ - y^+\| \le \|x - y\|$ for all $x, y \in \mathbb{R}^m$, from (12) it easy to conclude that

$$\|F(x) - F(y)\| \le \|A^\top A - I\|\|x - y\|, \quad \forall\ x, y \in \mathbb{R}^m.$$

Therefore, as by assumption $\|A^\top A - I\| < 1$ we may apply Theorem 1 with $\mathbb{X} = \mathbb{R}^m$, $T = F$, $d(x, y) = \|y - x\|$ for all $x, y \in \mathbb{R}^m$ and $\alpha = \|A^\top A - I\|$, for concluding that the Picard's Method (8) or equivalently, the sequence

$$x_{k+1} = F(x_k), \qquad k = 0, 1, \dots.$$

converges to a unique fixed point u of F, which from (12) is the solution of the Problem 2, i.e.,

$$\left(A^\top A - I\right) u^+ + u = A^\top z,$$

and by using Lemma 3 we have $P_K(z) = Au^+$. Moreover, Theorem 1 implies that the inequalities (10) and (11) hold. □

4.2 Picard's Method for solving Problem 3

The *Picard's method* for solving Problem 3 is formally defined by

$$\left(A^\top A+I\right)x_{k+1} = -\left(A^\top A-I\right)|x_k| + 2A^\top z, \qquad k=0,1,2,\ldots. \tag{13}$$

The sequence $\{x_k\}$ with starting point $x_0 \in \mathbb{R}^m$, called the *Picard's sequence* for solving equation (7) or for projecting a point $z \in \mathbb{R}^m$ onto the simplicial cone K. From now on we will refer this method as *Picard 2*.

Since $A \in \mathbb{R}^{m\times m}$ is a nonsingular matrix we conclude that $A^\top A$ is symmetric and positive definite. Hence, $A^\top A + I$ is nonsingular. Then for simplifying the notations define

$$C := \left(A^\top A+I\right)^{-1}\left(A^\top A-I\right). \tag{14}$$

Let $\lambda_1,\ldots\lambda_m$ and $\sigma_1,\ldots\sigma_m$ be the eigenvalues of $A^\top A$ and C, respectively. As $\lambda_i > 0$, for $i=1,2,\ldots m$, it easy to conclude that

$$\|C\| = \max\{|\sigma_1|,\ldots|\sigma_m|\} < 1, \quad \text{where} \qquad \sigma_i = \frac{1-\lambda_i}{\lambda_i+1}, \qquad i=1,2,\ldots m.$$

The next theorem provides the convergence of the Picard's iteration (13).

Theorem 4 *Let $A \in \mathbb{R}^{m\times m}$ be a nonsingular matrix, $K = A\mathbb{R}^m_+$ the corresponding simplicial cone and $z \in \mathbb{R}^m$ arbitrary. The Picard's sequence $\{x_k\}$ for solving Problem 3 is well defined and converges to the unique solution u of equation* (7) *from any starting point $x_0 \in \mathbb{R}^m$, $P_K(z) = Au^+$ and the following error bound holds*

$$\|u - x_k\| \le \frac{\|C\|}{1-\|C\|}\|x_k - x_{k-1}\|, \qquad \forall\, k=1,2\ldots. \tag{15}$$

Moreover, the sequence $\{x_k\}$ converges linearly to u as follows

$$\|u - x_{k+1}\| \le \|C\|\|u - x_k\|, \qquad k=0,1,2,\ldots. \tag{16}$$

Proof. Since the matrix $A^\top A+I$ is nonsingular, the function $F:\mathbb{R}^m \to \mathbb{R}^m$,

$$F(x) := -\left(A^\top A+I\right)^{-1}\left(A^\top A-I\right)|x| + 2\left(A^\top A+I\right)^{-1}A^\top z, \tag{17}$$

is well defined. Since $\||x|-|y|\| \le \|x-y\|$ for all $x,y \in \mathbb{R}^m$, from (17) and (14) we conclude that

$$\|F(x)-F(y)\| \le \|C\|\|x-y\|, \quad \forall\, x,y \in \mathbb{R}^m.$$

Therefore, as $\|C\| < 1$ we may apply Theorem 1 with $\mathbb{X} = \mathbb{R}^m$, $T = F$, $d(x,y) = \|y-x\|$ for all $x,y \in \mathbb{R}^m$ and $\alpha = \|C\|$, for concluding that the Picard's Method (13) or equivalently, the sequence

$$x_{k+1} = F(x_k), \qquad k=0,1,\ldots.$$

converges to a unique fixed point u of F, which from (17) is the solution of the Problem 3, i.e.,

$$\left(A^\top A+I\right)u + \left(A^\top A-I\right)|u| = 2A^\top z,$$

and by using Lemma 3 we have $P_K(z) = Au^+$. Moreover, Theorem 1 implies that the inequalities (15) and (16) hold. $\square$

5 Comparison between Picard's and Newton's methods

In this section theoretical and numerical comparisons of above Picard's methods and semi-smooth Newton's method studied in [17] are provided. Also Picard's method (13) is applied to solve an specific example.

5.1 Theoretic comparison

In this section theoretical comparisons between Picard's methods and semi-smooth Newton's method for solving Problem 1 will be provided.

It is shown in [17] that the semi-smooth Newton method applied to the equations (6), namely,

$$\left(\left(A^\top A - I\right) \operatorname{diag}(\operatorname{sgn}(x_k^+)) + I\right) x_{k+1} = A^\top z, \qquad k = 0, 1, 2, \ldots, \tag{18}$$

is always well defined and under the assumption

$$\|A^\top A - I\| < b < \frac{1}{3}, \tag{19}$$

on the matrix A defining the simplicial cone $K = A\mathbb{R}^m_+$, the generated sequence $\{x_k\}$ converges linearly to the unique solution u of Problem 2 from any starting point and, as a consequence of Lemma 3 we have $P_K(z) = Au^+$ for any $z \in \mathbb{R}^m$, which implies that u solves Problem 1.

Problem 1, i.e., the problem of projecting a point $z \in \mathbb{R}^m$ onto a simplicial cone $K = A\mathbb{R}^m_+$ is equivalent, by Lemma 3, to solving either Problem 2 or Problem 3. Note that for solving Problems 2 by Picard's method (8) assumption (9) on the matrix A (see Theorem 3) is less restrictive than assumption (19). When solving Problem 3 we only need the invertibility of the matrix A for Picard's method (13) to converge (see Theorem 4). Therefore, Picard's method (13) is theoretically more robust than Picard's method (8) and consequently than semi-smooth Newton method (18). In the next section we will present an example, where according to the established theory, only Picard's method (13) can be applied.

The main drawbacks of Picard (13) and semi-smooth Newton (18) is that both require the solution of a linear system in each iteration which constitute the largest computational effort of these methods. Picard's method (8) do not have to solve a linear system, avoiding more complicated calculations, which is particularly interesting for large scale problems. We will investigate the efficiency of these method in section 5.2.1.

5.1.1 Example

Consider the monotone nonnegative cone, which is a simplicial cone K defined by

$$K := \left\{x = (x^1, \ldots, x^m) \in \mathbb{R}^m,\ x^1 \geq x^2 \geq \ldots x^m \geq 0\right\}, \tag{20}$$

The monotone nonnegative cone and the projection onto it occurs in various important practical problems such as the problem of map-making from relative distance information e.g., stellar cartography (see `www.convexoptimization.com/wikimization/index.php/Projection_on_Polyhedral_Convex_Cone` and Section 5.13.2 in [12]) and isotonic regression [8, 21, 22, 36]. The isotonic regression

[3, 7, 39, 40] is a very important topic in statistics with hundreds of papers and several books dedicated to this topic. This section provides a different view about projecting onto the monotone nonnegative cones via Picard's method (13) which is related to the iterative theory of bidiagonal and tridiagonal matrices, and the Fibonacci numbers. The dual of the monotone nonnegative cone is $K^* = A\mathbb{R}^m_+$, where $A \in \mathbb{R}^{m\times m}$ is the nonsingular matrix

$$A = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}, \qquad A^\top A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}.$$

From Remark 3, the problem of projecting a point onto K^* is equivalent to projecting onto K. Let $\lambda_1, \dots \lambda_m$ be the eigenvalues of $A^\top A$. From [47] we have that the eigenvalues of matrix $A^\top A$ are given by

$$\lambda_i = 2 + 2\cos\left(\frac{2i\pi}{2m+1}\right), \qquad i = 1, 2, \dots m. \tag{21}$$

Hence from (21) we conclude that

$$0 < \lambda_i < 4, \qquad \lim_{m\to\infty} \lambda_m = 0, \qquad \lim_{m\to\infty} \lambda_1 = 4, \qquad \lim_{m\to\infty} \|A^\top A - I\| = 3.$$

Since $\|A^\top A - I\| > 1$ for all $m \geq 2$, for projecting a point onto the cone K^*, we can not apply semi-smooth Newton method studied in [17] neither Picard's iteration (8). However Picard's method (13) can be used. In order to reduce the computational cost of this method, for solving the linear system involved in each iteration, we suggest the following triangular decomposition

$$A^\top A + I = \begin{pmatrix} d_1 & -1 & & & \\ & d_2 & -1 & & \\ & & \ddots & \ddots & \\ & & & d_{m-1} & -1 \\ & & & & d_m \end{pmatrix} \begin{pmatrix} 1 & & & & \\ -\frac{1}{d_2} & 1 & & & \\ & -\frac{1}{d_3} & \ddots & & \\ & & \ddots & 1 & \\ & & & -\frac{1}{d_m} & 1 \end{pmatrix},$$

where $d_m = 2$, $d_i = 3 - 1/d_{i+1}$ for $i = m-1, \cdots, 1$. Another alternative for solving the linear system would be to compute the matrices

$$R = \left(A^\top A + I\right)^{-1}\left(A^\top A - I\right), \qquad S = \left(A^\top A + I\right)^{-1} A^\top.$$

By using the recursion formulas for a tridiagonal matrix from [11], which are based on the results of [18, 45, 46], after some algebraic manipulations and taking into account that R is symmetric we obtain

$$R_{ij} = \begin{cases} -\dfrac{2F_{2i}F_{2m-2j+1}}{F_{2m+1}} & \text{if} \quad 1 < i < j < m, \\[2ex] \dfrac{F_{2i}F_{2m-2i} - F_{2i-2}F_{2m-2i+1}}{F_{2m+1}} & \text{if} \quad 1 < i = j < m, \\[2ex] \dfrac{2F_{2m-2j+1}}{F_{2m+1}} & \text{if} \quad 1 = i < j < m, \\[2ex] -\dfrac{2F_{2i}}{F_{2m+1}} & \text{if} \quad 1 < i < j = m, \\[2ex] -\dfrac{2}{F_{2m+1}} & \text{if} \quad i = 1,\ j = m, \\[2ex] \dfrac{F_{2m-2}}{F_{2m+1}} & \text{if} \quad i = j = 1, m. \end{cases}$$

$$S_{ij} = \begin{cases} -\dfrac{F_{2i}F_{2m-2j+2}}{F_{2m+1}} & \text{if} \quad i < j, \\[2ex] \dfrac{F_{2j-1}F_{2m-2i+1}}{F_{2m+1}} & \text{if} \quad 1 < j \le i, \\[2ex] \dfrac{F_{2m-2i+1}}{F_{2m+1}} & \text{if} \quad 1 = j \le i. \end{cases}$$

where F_i is the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_{i+2} = F_i + F_{i+1}$.

5.2 Computational results

In this section we present two numerical experiments. In the first, numerical comparisons between Picard's methods (8), (13) and semi-smooth Newton's method (18) for solving Problem 1 will be provided. In the second one, we study the behavior Picard's method (13) solving the problem described in Section 5.1.1. All programs were implemented in MATLAB Version 7.11 64-bit and run on a $3.40GHz$ Intel Core $i5 - 4670$ with $8.0GB$ of RAM. All MATLAB codes and generated data of this paper are available in `http://orizon.mat.ufg.br/pages/34449-publications`.

General considerations:

- In order to accurately measure the method's runtime for a problem, each of them was solved 10 times and the runtime data collected. Then, we defined the corresponding *method's runtime for a problem* as the median of these measurements.
- We consider that the method converged to the solution and stopped the execution when, for some k, the condition

$$\frac{\|u - x_k\|}{\|u\|} < RelativeTolerance,$$

is satisfied.

5.2.1 Numerical experiment I

In this experiment, we study the percentage of problems for which a method was the fastest one (**efficiency**) to compare them. With the aim that methods (8), (13) and (18) find solution on 1000 generated random test problems of dimension $m = 1000$, we construct the matrix A (defining the simplicial cone $K = A\mathbb{R}^m_+$) in each problem satisfying the condition (19).

We assume that a method is the fastest one for a problem, if the corresponding runtime is less than or equal than 1.01 times *the best time* of all methods to find the solution.

Each test problem was generated as follows:

(i) To construct the matrix $A \in \mathbb{R}^{m\times m}$ satisfying the condition (19), we first chose a random number b from the standard uniform distribution on the open interval $(0, 1/3)$. Then, we chose a random number $\bar{b}$ from the standard uniform distribution on the open interval $(0, b)$. We compute the matrices S, V and D, respectively, from the singular value decomposition of a $m \times m$ generated real matrix containing random values drawn from the uniform distribution on the interval $[-10^6, 10^6]$. Finally we computed

$$A = S(sqrt(I + \bar{b}(V/v)))D,$$

were v is the largest singular value of V, the operator / denotes element-by-element division and $sqrt(M)$ is the square root of the elements of the matrix M.

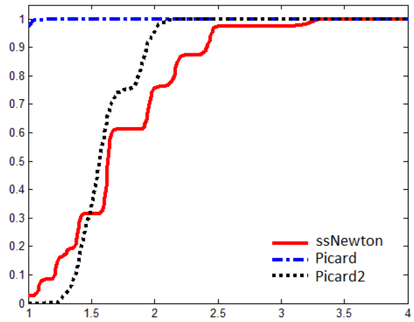
(ii) We chose the solution $u \in \mathbb{R}^m$ containing random values drawn from the uniform distribution on the interval $[-10^6, 10^6]$ and computed $z \in \mathbb{R}^m$ from equation (6). Finally we chose a starting point $x_0 \in \mathbb{R}^m$ containing random values drawn from the uniform distribution on the interval $[-10^6, 10^6]$.

In order to provide information for the analysis of the large test problems set considered, we use the performance profiles (see [14]). The performance profiles for a method is the cumulative distribution function for a performance metric. In this case we use the ratio of the method's runtime versus the best runtime of all of the methods as the performance metric. **Efficiency** can be checked in the value of the profile function at 1.

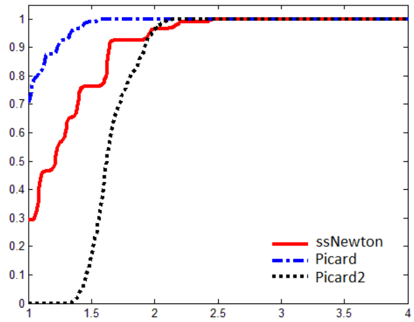
Figure 1 shows the performance profiles of the three methods for different relative tolerance values. These graphs reveal that Picard's method (13) was the most efficient for low and medium accuracy, while semi-smooth Newton's method (18) was the most efficient for high accuracy requirements. However, since semi-smooth Newton's method (18) requires at each step the solution of a system of linear equations, which may become unreasonably expensive computationally as the problem dimension increases, these results suggest that for large scale problems Picard's method (8) is recommended.

On the other hand, Picard's method (13) was always the worst, except in the low accuracy case. It can be inferred from Figure 2, where convergence mean time for each problem consumed by Picard's method (13) is less than consumed by semi-smooth Newton's method (18).

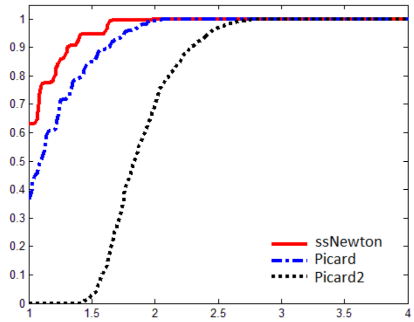
Figure 2 shows, as one would expect, the number of iteration on semi-smooth Newton method is less than Picard's methods (8) and (13) for solving the same set of problems and only from certain tolerance semi-smooth Newton method consume less time.



(a) RelativeTolerance=10^{-7}



(b) RelativeTolerance=10^{-10}



(c) RelativeTolerance=10^{-13}

Figure 1: Performance profiles on [1,4] for different accuracies. ssNewton, Picard and Picard2 denotes the methods (8), (13) and (18), respectively.

5.2.2 Numerical experiment II

In this experiment, we study the behavior of Picard's method (13) solving the problem described in Section 5.1.1 on sets of 100 generated random test problems of dimension $m = 100$, $m = 500$, $m = 1000$, $m = 1500$, $m = 2000$, respectively.

Each $m-$dimensional test problem was generated as follows: We construct the matrix A (defining the simplicial cone $K^* = A\mathbb{R}^m_+$) as is defined in Section 5.1.1. We chose the solution $u \in \mathbb{R}^m$, computed $z \in \mathbb{R}^m$ and chose a starting point $x_0 \in \mathbb{R}^m$ as we described in the previous Section 5.2.1.

The computational results obtained are reported in Table 1. From these, it can be noted that for the same dimension, to achieve higher accuracy, the method needs to perform a greater number of iterations and consequently consume more runtime. The same behavior occurs when, for the same accuracy, the dimension of the problem increases.

Dimension	Total Iterations			Total Time		
$m = 100$	4927	7475	10036	1.096	1.624	2.180
$m = 500$	6613	10333	14055	66.183	103.411	140.812
$m = 1000$	8120	12873	17640	449.507	717.310	984.274
$m = 1500$	8159	12924	17732	1358.698	2151.743	2952.247
$m = 2000$	8814	14054	19359	3098.215	4955.041	6820.121
Relative Tolerance	10^{-7}	10^{-10}	10^{-13}	10^{-7}	10^{-10}	10^{-13}

Table 1: Total overall iterations and total time in seconds, performed and consumed, respectively by Picard's method (13) to solve the 100 test problems of each dimension for different accuracies.

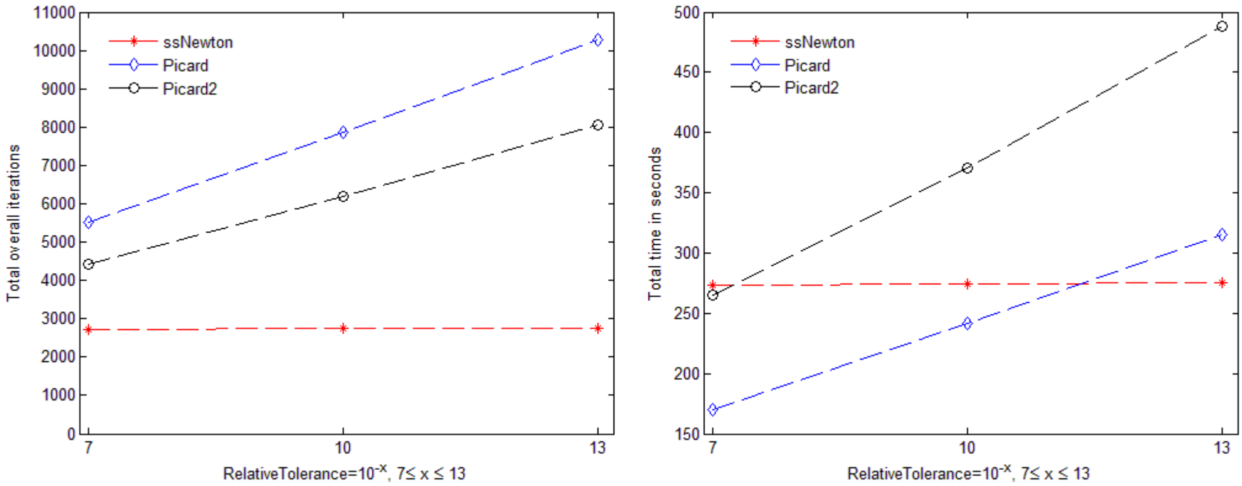


Figure 2: Total overall iterations and total time in seconds, performed and consumed, respectively by each method to solve the 1000 test problems for different accuracies. ssNewton, Picard and Picard2 denotes the methods (8), (13) and (18), respectively.

6 Conclusions

In this paper we studied the problem of projection onto a simplicial cone which, via Moreau's decomposition theorem for projecting onto cones, is reduced to finding the unique solution of a nonsmooth system of equations. Our main results show that, under a mild assumption on the simplicial cone, we can apply Picard's method for finding a unique solution of the obtained associated system and that the generated sequence converges linearly to the solution for any starting point. Note that in Theorem 4 we do not make any assumption on the simplicial cone, on the other hand, we have to solve a linear equation in each iteration. It would be interesting to see whether the used technique can be applied for finding the projection onto more general cones. As has been shown in [44], the problem of projection onto a simplicial cone is reduced to a certain type of linear complementarity problem (LCP). Numerical comparisons between Picard's methods (8,13) and semi-smooth Newton's method (18) for solving Problem 1 was provided in Section 5. It would also be interesting to compare these methods with the methods proposed in [16, 32, 44] and the Lemke's method for LCPs.

References

[1] M. Abbas and S. Z. Németh. Solving nonlinear complementarity problems by isotonicity of the metric projection. *J. Math. Anal. Appl.*, 386(2):882–893, 2012.

[2] K. S. Al-Sultan and K. G. Murty. Exterior point algorithms for nearest points and convex quadratic programs. *Math. Programming*, 57(2, Ser. B):145–161, 1992.

[3] R. E. Barlow, D. J. Bartholomew, J. M. Bremner, and H. D. Brunk. *Statistical inference under order restrictions. The theory and application of isotonic regression.* John Wiley & Sons, London-New York-Sydney, 1972. Wiley Series in Probability and Mathematical Statistics.

[4] H. H. Bauschke and J. M. Borwein. On projection algorithms for solving convex feasibility problems. *SIAM Rev.*, 38(3):367–426, 1996.

[5] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty. *Nonlinear programming.* Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, third edition, 2006. Theory and algorithms.

[6] R. Berk and R. Marcus. Dual cones, dual norms, and simultaneous inference for partially ordered means. *J. Amer. Statist. Assoc.*, 91(433):318–328, 1996.

[7] M. J. Best and N. Chakravarti. Active set algorithms for isotonic regression; an unifying framework. *Math. Programming*, 47:425–439, 1990.

[8] O. Burdakov, I. Kapyrin, and Y. Vassilevski. Monotonicity recovering and accuracy preserving optimization methods for postprocessing finite element solutions. *J. Comput. Phys.*, 231(8):3126–3142, 2012.

[9] Y. Censor, T. Elfving, G. T. Herman, and T. Nikazad. On diagonally relaxed orthogonal projection methods. *SIAM J. Sci. Comput.*, 30(1):473–504, 2007/08.

[10] Y. Censor, D. Gordon, and R. Gordon. Component averaging: an efficient iterative parallel algorithm for large and sparse unstructured problems. *Parallel Comput.*, 27(6):777–808, 2001.

[11] C. M. da Fonseca. On the eigenvalues of some tridiagonal matrices. *J. Comput. Appl. Math.*, 200(1):283–286, 2007.

[12] J. Dattorro. *Convex Optimization and Euclidean Distance Geometry.* M$\varepsilon\beta$oo publishing, v2014.04.08, 2014.

[13] F. Deutsch and H. Hundal. The rate of convergence of Dykstra's cyclic projections algorithm: the polyhedral case. *Numer. Funct. Anal. Optim.*, 15(5-6):537–565, 1994.

[14] E. D. Dolan and J. J. Moré. Benchmarking optimization software with performance profiles. *Math. Programming*, 91(2):201–213, 2002.

[15] R. L. Dykstra. An algorithm for restricted least squares regression. *J. Amer. Statist. Assoc.*, 78(384):837–842, 1983.

[16] A. Ekárt, A. B. Németh, and S. Z. Németh. Rapid heuristic projection on simplicial cones, 2010, arXiv:1001.1928.

[17] O. P. Ferreira and S. Z. Németh. Projection onto simplicial cones by a semi-smooth Newton method. *Optimization Letters*, http://dx.doi.org/10.1007/s11590-014-0775-1, Aug. 2014.

[18] C. F. Fischer and R. A. Usmani. Properties of some tridiagonal matrices and their application to boundary value problems. *SIAM J. Numer. Anal.*, 6:127–142, 1969.

[19] J. D. Foley, A. van Dam, S. K. Feiner, and J. F. Hughes. *Computer Graphics: Principles and Practice.* Addison-Wesley systems programming series, 1990.

[20] H. Frick. Computing projections into cones generated by a matrix. *Biometrical J.*, 39(8):975–987, 1997.

[21] S. J. Grotzinger and C. Witzgall. Projections onto order simplexes. *Appl. Math. Optim.*, 12(3):247–270, 1984.

[22] A. Guyader, N. Jégou, A. B. Németh, and S. Z. Németh. A geometrical approach to iterative isotone regression. *Appl. Math. Comput.*, 227:359–369, 2014.

[23] J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex analysis and minimization algorithms: Fundamentals. I*, volume 305 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1993.

[24] X. Hu. An exact algorithm for projection onto a polyhedral cone. *Aust. N. Z. J. Stat.*, 40(2):165–170, 1998.

[25] T. Huynh, C. Lassez, and J.-L. Lassez. Practical issues on the projection of polyhedral sets. *Ann. Math. Artificial Intelligence*, 6(4):295–315, 1992. Artificial intelligence and mathematics, II.

[26] G. Isac and A. B. Németh. Monotonicity of metric projections onto positive cones of ordered Euclidean spaces. *Arch. Math. (Basel)*, 46(6):568–576, 1986.

[27] G. Isac and A. B. Németh. Isotone projection cones in Euclidean spaces. *Ann. Sci. Math. Québec*, 16(1):35–52, 1992.

[28] E. Kreyszig. *Introductory functional analysis with applications.* John Wiley & Sons, New York-London-Sydney, 1978.

[29] Z. Liu and Y. Fathi. An active index algorithm for the nearest point problem in a polyhedral cone. *Comput. Optim. Appl.*, 49(3):435–456, 2011.

[30] Z. Liu and Y. Fathi. The nearest point problem in a polyhedral set and its extensions. *Comput. Optim. Appl.*, 53(1):115–130, 2012.

[31] J. J. Moreau. Décomposition orthogonale d'un espace hilbertien selon deux cônes mutuellement polaires. *C. R. Acad. Sci.*, 255:238–240, 1962.

[32] P. M. Morillas. Dykstra's algorithm with strategies for projecting onto certain polyhedral cones. *Appl. Math. Comput.*, 167(1):635–649, 2005.

[33] K. G. Murty. *Linear complementarity, linear and nonlinear programming*, volume 3 of *Sigma Series in Applied Mathematics.* Heldermann Verlag, Berlin, 1988.

[34] K. G. Murty and Y. Fathi. A critical index algorithm for nearest point problems on simplicial cones. *Math. Programming*, 23(2):206–215, 1982.

[35] A. B. Németh and S. Z. Németh. How to project onto an isotone projection cone. *Linear Algebra Appl.*, 433(1):41–51, 2010.

[36] A. B. Németh and S. Z. Németh. How to project onto the monotone nonnegative cone using pool adjacent violators type algorithms, 2012, arXiv:1201.2343.

[37] S. Z. Németh. Characterization of latticial cones in Hilbert spaces by isotonicity and generalized infimum. *Acta Math. Hungar.*, 127(4):376–390, 2010.

[38] S. Z. Németh. Isotone retraction cones in Hilbert spaces. *Nonlinear Anal.*, 73(2):495–499, 2010.

[39] P. S. Puri and H. Singh. On recursive formulas for isotonic regression useful for statistical inference under order restrictions. *J. Statist. Plann. Inference*, 24(1):1–11, 1990.

[40] T. Robertson, F. T. Wright, and R. L. Dykstra. *Order restricted statistical inference.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Ltd., Chichester, 1988.

[41] H. D. Scolnik, N. Echebest, M. T. Guardarucci, and M. C. Vacchino. Incomplete oblique projections for solving large inconsistent linear systems. *Math. Program.*, 111(1-2, Ser. B):273–300, 2008.

[42] G. W. Stewart. On the perturbation of pseudo-inverses, projections and linear least squares problems. *SIAM Rev.*, 19(4):634–662, 1977.

[43] M. Tan, G.-L. Tian, H.-B. Fang, and K. W. Ng. A fast EM algorithm for quadratic optimization subject to convex constraints. *Statist. Sinica*, 17(3):945–964, 2007.

[44] M. Ujvári. On the projection onto a finitely generated cone, 2007, Preprint WP 2007-5, MTA SZTAKI, Laboratory of Operations Research and Decision Systems, Budapest.

[45] R. A. Usmani. Inversion of a tridiagonal Jacobi matrix. In *Proceedings of the 3rd ILAS Conference (Pensacola, FL, 1993)*, volume 212/213, pages 413–414, 1994.

[46] R. A. Usmani. Inversion of Jacobi's tridiagonal matrix. *Comput. Math. Appl.*, 27(8):59–66, 1994.

[47] A. R. Willms. Analytic results for the eigenvalues of certain tridiagonal matrices. *SIAM J. Matrix Anal. Appl.*, 30(2):639–656, 2008.

[48] S. Xu. Estimation of the convergence rate of Dykstra's cyclic projections algorithm in polyhedral case. *Acta Math. Appl. Sinica (English Ser.)*, 16(2):217–220, 2000.